

Homoclinic tangencies and singular hyperbolicity for three-dimensional vector fields

Sylvain Crovisier

Dawei Yang *

February 2017

Abstract

We prove that any vector field on a three-dimensional compact manifold can be approximated in the C^1 -topology by one which is singular hyperbolic or by one which exhibits a homoclinic tangency associated to a regular hyperbolic periodic orbit. This answers a conjecture by Palis [Pa₂].

During the proof we obtain several other results with independent interest: a compactification of the rescaled sectional Poincaré flow and a generalization of Mañé-Pujals-Sambarino theorem for three-dimensional C^2 vector fields with singularities.

Contents

| | | |
|---|--|----|
| 1 | Introduction | 1 |
| 2 | Compactification of the sectional flow | 7 |
| 3 | Fibered dynamics with a dominated splitting | 15 |
| 4 | Topological hyperbolicity | 34 |
| 5 | Markovian boxes | 56 |
| 6 | Uniform hyperbolicity | 71 |
| 7 | Generalized Mañé-Pujals-Sambarino theorem for singular flows | 81 |
| 8 | C^1 -generic three-dimensional vector fields | 86 |

1 Introduction

1.1 Homoclinic tangencies and singular hyperbolicity

A main problem in differentiable dynamics is to describe a class of systems as large as possible. This approach started in the 60's with the theory of *hyperbolic systems* introduced by Smale and Anosov, among others. A flow $(\varphi_t)_{t \in \mathbb{R}}$ on a manifold M , generated by a vector field X , is hyperbolic if its chain-recurrent set (defined in [Co]) is the finite union of invariant sets Λ that are hyperbolic: each one is endowed with an invariant splitting into continuous sub-bundles

$$TM|_{\Lambda} = E^s \oplus (\mathbb{R}X) \oplus E^u$$

such that E^s (resp. E^u) is uniformly contracted by $D\varphi_T$ (resp. $D\varphi_{-T}$) for some $T > 0$. The dynamics of these systems has been deeply described.

The set of hyperbolic vector fields is open and dense in the space $\mathcal{X}^r(M)$ of C^r -vector fields when M is an orientable surface and $r \geq 1$ [Pe] or when M is an arbitrary surface and $r = 1$ [Pu].

*S.C was partially supported by the ANR project *DynNonHyp* BLAN08-2 313375, by the Balzan Research Project of J. Palis and by the ERC project 692925 *NUHGD*. D.Y. was partially supported by NSFC 11271152, ANR project *DynNonHyp* BLAN08-2313375 and and A Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions(PAPD).

Smale raised the problem of the abundance of hyperbolicity for higher dimensional manifolds. Newhouse's work [Ne] for surface diffeomorphisms implies that hyperbolicity is not dense in the spaces $\mathcal{X}^r(M)$, $r \geq 2$, once the dimension of M is larger or equal to three. Indeed a bifurcation called *homoclinic tangency* leads to a robust phenomenon for C^2 -vector fields which is expected to be one of the main obstructions to hyperbolicity. A vector field X has a homoclinic tangency if there exist a hyperbolic non-singular periodic orbit γ and an intersection x of the stable and unstable manifolds of γ which is not transverse (i.e. $T_x W^s(\gamma) + T_x W^u(\gamma) \neq T_x M$). It produces rich wild behaviors.

The flows with singularities may have completely different dynamics. The class of three-dimensional vector fields contains a very early example of L. N. Lorenz [Lo], that he called “butterfly attractors”. Trying to understand this example, some robustly non-hyperbolic attractors (which are called “geometrical Lorenz attractors”) were constructed by [ABS, G, GW]. The systems cannot be accumulated by vector fields with homoclinic tangencies. While they are less wild than systems in the Newhouse domain, this defines a new class of dynamics, where the lack of hyperbolicity is related to the presence of a singularity.

Morales, Pacifico and Pujals [MPP] have introduced the notion of *singular hyperbolicity* to characterize these Lorenz-like dynamics. A compact invariant set Λ is called singular hyperbolic if either X or $-X$ satisfies the following property. There exists an invariant splitting into continuous sub-bundles

$$TM|_{\Lambda} = E^s \oplus E^{cu}$$

and a constant $T > 0$ such that:

- domination: $\forall x \in \Lambda, u \in E^s(x) \setminus \{0\}, v \in E^{cu}(x) \setminus \{0\}, \frac{\|D\varphi_T \cdot u\|}{\|u\|} \leq 1/2 \frac{\|D\varphi_T \cdot v\|}{\|v\|},$
- contraction: $\forall x \in \Lambda, \|D\varphi_T|_{E^s}(x)\| \leq 1/2,$
- sectional expansion: $\forall x \in \Lambda, \forall P \in \text{Gr}_2(E^{cu}(x)), |\text{Jac}(D\varphi_{-T}|_P)| \leq 1/2.$

When $\Lambda \cap \text{Sing}(X) = \emptyset$ this notion coincides with the hyperbolicity. A flow is *singular hyperbolic* if its chain-recurrent set is a finite union of singular-hyperbolic sets. This property defines an open subset in the space of C^1 vector fields. Such flow has good topological and ergodic properties, see [AP]. Note that hyperbolicity implies singular hyperbolicity by this definition.

Palis [Pa₁, Pa₂, Pa₃, Pa₄] formulated conjectures for typical dynamics of diffeomorphisms and vector fields. He proposed that homoclinic bifurcations and Lorenz-like dynamics are enough to characterize the non-hyperbolicity. For three-dimensional manifolds, this is more precise (see also [BDV, Conjecture 5.14]):

Conjecture (Palis). *For any $r \geq 1$ and any three dimensional manifold M , every vector field in $\mathcal{X}^r(M)$ can be approximated by one which is hyperbolic, or by one which display a homoclinic tangency, a singular hyperbolic attractor or a singular hyperbolic repeller.*

In higher topologies $r > 1$, such a general statement is for now out of reach, but more techniques have been developed in the C^1 -topology [Cr₂]. This allows us to prove the conjecture above for $r = 1$. This has been announced in [CY₁].

Main Theorem. *On any three-dimensional compact manifold M , any C^1 vector field can be approximated in $\mathcal{X}^1(M)$ by singular hyperbolic vector fields, or by ones with homoclinic tangencies.*

An important step towards this result was the dichotomy between hyperbolicity and homoclinic tangencies for surface diffeomorphisms by Pujals and Sambarino [PS₁]. Arroyo and Rodriguez-Hertz then obtained [ARH] a version of the theorem above for vector fields without singularities. The main difficulty of the present paper is to address the existence of singularities.

The Main Theorem allows to extend Smale's spectral theorem for the C^1 -generic vector fields far from homoclinic tangencies. Recall that an invariant compact set Λ is *robustly transitive* for a vector field X if there exists a neighborhood U of Λ and $\mathcal{U} \subset \mathcal{X}^1(M)$ of X such that, for any $Y \in \mathcal{U}$, the maximal invariant set of Y in U is transitive (i.e. admits a dense forward orbit).

Corollary 1.1. *If $\dim(M) = 3$, there exists a dense open subset $\mathcal{U} \subset \mathcal{X}^1(M)$ such that, for any vector field $X \in \mathcal{U}$ which can not be approximated by one exhibiting a homoclinic tangency, the chain-recurrent set is the union of finitely many robustly transitive sets.*

When $\dim(M) = 3$, there exists Newhouse domains in $\mathcal{X}^r(M)$, $r \geq 2$. But we note that there is no example of a non empty open set $\mathcal{U} \subset \mathcal{X}^1(M)$ such that homoclinic tangencies occur on a dense subset of \mathcal{U} . This raises the following conjecture.

Conjecture. *If $\dim(M) = 3$, any vector field can be approximated in $\mathcal{X}^1(M)$ by singular hyperbolic ones.*

Even for non-singular vector fields, the conjecture above is open. It claims the density of hyperbolicity and it has a counterpart for surface diffeomorphisms, sometimes called Smale's conjecture.

The chain-recurrent set naturally decomposes into invariant compact subsets that are called *chain-recurrence classes* (see [Co] and Section 8.1). The conjecture holds if one shows that for C^1 -generic vector fields, any chain-recurrence class has a dominated splitting (see Theorem 8.4 below). An important case would be to rule out for C^1 -generic vector fields the existence of non-trivial chain-recurrence classes containing a singularity with a complex eigenvalue.

Note that the conjecture also asserts that for typical 3-dimensional vector fields, the non-trivial singular behaviors only occur inside Lorenz-like attractors and repellers.

1.2 Dominated splittings in dimension 3

The first step for proving the hyperbolicity or the singular hyperbolicity is to get a dominated splitting for the tangent flow $D\varphi$. For surface diffeomorphisms far from homoclinic tangencies, this has been proved in [PS₁]. For vector fields, it is in general much more delicate. Indeed one has to handle with sets which may contain both regular orbits (for which $\mathbb{R}X$ is a non-degenerate invariant sub-bundle) and singularities: for instance it is not clear how to extend the tangent splittings at a singularity and along its stable and unstable manifolds.

Since the flow direction does not see any hyperbolicity, it is fruitful to consider another linear flow that has been defined by Liao [Li₁]. To each vector field X , one introduces the singular set $\text{Sing}(X) = \{\sigma \in M : X(\sigma) = 0\}$ and the *normal bundle* \mathcal{N} which is the collection of subspaces $\mathcal{N}_x = \{v \in T_x M : \langle X(x), v \rangle = 0\}$ for $x \in M \setminus \text{Sing}(X)$. One then defines the *linear Poincaré flow* $(\psi_t)_{t \in \mathbb{R}}$ by projecting orthogonally on \mathcal{N} the tangent flow:

$$\psi_t(v) = D\varphi_t(v) - \frac{\langle D\varphi_t(v), X(\varphi_t(x)) \rangle}{\|X(\varphi_t(x))\|^2} X(\varphi_t(x)).$$

If $\Lambda \subset M$ is a (not necessarily compact) invariant set, an invariant continuous splitting $TM|_\Lambda = E \oplus F$ is dominated if it satisfies the first item of the definition of singular hyperbolicity stated above. We also say that the linear Poincaré flow over $\Lambda \setminus \text{Sing}(X)$ admits a *dominated splitting* when there exists a continuous invariant splitting $\mathcal{N}|_{\Lambda \setminus \text{Sing}(X)} = \mathcal{E} \oplus \mathcal{F}$ and a constant $T > 0$ such that

$$\forall x \in \Lambda \setminus \text{Sing}(X), u \in \mathcal{E}(x) \setminus \{0\}, v \in \mathcal{F}(x) \setminus \{0\}, \quad \frac{\|\psi_{T \cdot u}\|}{\|u\|} \leq \frac{1}{2} \frac{\|\psi_T \cdot v\|}{\|v\|}.$$

A dominated splitting on Λ for the tangent flow $D\varphi$ always extends to the closure of Λ : for that reason, one usually considers compact sets. But the dominated splittings of the linear Poincaré flow can not always be extended to the closure of the invariant set Λ since the closure of Λ may contain singularities, where the linear Poincaré flow is not defined. It is however natural to consider the linear Poincaré flow: for vector fields away from systems exhibiting a homoclinic tangency, the natural splitting of a hyperbolic saddles is dominated for ψ (see [GY]), but this is not the case in general for $D\varphi$. In particular the existence of dominated splitting for the linear Poincaré flow does not imply the existence of a dominated splitting for the tangent flow.

However the equivalence between these two properties holds for C^1 -generic vector fields on chain-transitive sets (whose definition is recalled in Section 8.1).

Theorem A. *When $\dim(M) = 3$, there exists a dense G_δ subset $\mathcal{G} \subset \mathcal{X}^1(M)$ such that for any $X \in \mathcal{G}$ and any chain-transitive set Λ (which is not reduced to a periodic orbit or a singularity), the linear Poincaré flow over $\Lambda \setminus \text{Sing}(X)$ admits a non-trivial dominated splitting if and only if the tangent flow over Λ does.*

The Main Theorem will then follow easily: as already mentioned, far from homoclinic tangencies, the linear Poincaré flow is dominated, hence the tangent flow is also. The singular hyperbolicity then follows from the domination of the tangent flow, as it was shown in [ARH] for chain-recurrence classes without singularities and in a recent work by Gan and Yang [GY] for the singular case.

Theorem A is a consequence of a similar result for (non-generic) C^2 vector fields.

Theorem A' (Equivalence between dominated splittings). *When $\dim M = 3$, Consider a C^2 vector field X on M with a flow φ and an invariant compact set Λ with the following properties:*

- *Any singularity $\sigma \in \Lambda$ is hyperbolic, has simple real eigenvalues; the smallest one is negative and its invariant manifold satisfies $W^{ss}(\sigma) \cap \Lambda = \{\sigma\}$.*
- *For any periodic orbit in Λ , the smallest Lyapunov exponent is negative.*
- *There is no subset of Λ which is a repeller supporting a dynamics which is the suspension of an irrational circle rotation.*

Then the tangent flow $D\varphi$ on Λ has a dominated splitting $TM|_\Lambda = E \oplus F$ with $\dim(E) = 1$ if and only if the linear Poincaré flow on $\Lambda \setminus \text{Sing}(X)$ has a dominated splitting.

1.3 Compactification of the normal flow

In this paper, we use techniques for studying flows that may be useful for other problems.

Local fibered flows. In order to analyze the tangent dynamics and to prove the existence of a dominated splitting over a set Λ , one needs to analyze the local dynamics near Λ . For a diffeomorphism f , one usually lifts the local dynamics to the tangent bundle: for each $x \in M$, one defines a diffeomorphism $\hat{f}_x: T_x M \rightarrow T_{f(x)} M$, which preserves the 0-section (i.e. $\hat{f}_x(0_x) = 0_{f(x)}$) and is locally conjugated to f through the exponential map. It defines in this way a local fibered system on the bundle $TM \rightarrow M$. For flows one introduces a similar notion.

Definition 1.2 (Local fibered flow). *Let $(\varphi_t)_{t \in \mathbb{R}}$ be a continuous flow over a compact metric space K , and let $\mathcal{N} \rightarrow K$ be a continuous Riemannian vector bundle. A local C^k -fibered flow P on \mathcal{N} is a continuous family of C^k -diffeomorphisms $P_t: \mathcal{N}_x \rightarrow \mathcal{N}_{\varphi_t(x)}$, for $(x, t) \in K \times \mathbb{R}$, preserving 0-section with the following property.*

There is $\beta_0 > 0$ such that for each $x \in K$, $t_1, t_2 \in \mathbb{R}$, and $u \in \mathcal{N}_x$ satisfying

$$\|P_{s,t_1}(u)\| \leq \beta_0 \text{ and } \|P_{s,t_2}(P_{t_1}(u))\| \leq \beta_0 \text{ for each } s \in [0, 1],$$

then we have

$$P_{t_1+t_2}(u) = P_{t_2} \circ P_{t_1}(u).$$

For a vector field X , a natural way to lift the dynamics is to define the *Poincaré map* by projecting the normal spaces \mathcal{N}_x and $\mathcal{N}_{\varphi_t(x)}$ above two points of a regular orbit using the exponential map¹. Then the Poincaré map P_t defines a local diffeomorphism from \mathcal{N}_x to $\mathcal{N}_{\varphi_t(x)}$. The advantage of this construction is that the dimension has been dropped by 1.

Extended flows. A new difficulty appears: the domain of the Poincaré maps degenerate near the singularities. For that reason one introduces the *rescaled sectional Poincaré flow*:

$$P_t^*(u) = \|X(\varphi_t(x))\|^{-1} \cdot P_t(\|X(x)\| \cdot u).$$

In any dimension this can be compactified as a fibered lifted flow, assuming that the singularities are not degenerate.

Theorem B (Compactification). *Let X be a C^k -vector field, $k \geq 1$, over a compact manifold M . Let $\Lambda \subset M$ be a compact set which is invariant by the flow $(\varphi_t)_{t \in \mathbb{R}}$ associated to X such that $DX(\sigma)$ is invertible at each singularity $\sigma \in \Lambda$.*

Then, there exists a topological flow $(\widehat{\varphi}_t)_{t \in \mathbb{R}}$ over a compact metric space $\widehat{\Lambda}$, and a local C^k -fibered flow (\widehat{P}_t^) over a Riemannian vector bundle $\widehat{\mathcal{N}\mathcal{M}} \rightarrow \widehat{\Lambda}$ whose fibers have dimension $\dim(M) - 1$ such that:*

- *the restriction of φ to $\Lambda \setminus \text{Sing}(X)$ embeds in $(\widehat{\Lambda}, \widehat{\varphi})$ through a map i ,*
- *the restriction of $\widehat{\mathcal{N}\mathcal{M}}$ to $i(\Lambda \setminus \text{Sing}(X))$ is isomorphic to the normal bundle $\mathcal{N}\mathcal{M}|_{\Lambda \setminus \text{Sing}(X)}$ through a map I , which is fibered over i and which is an isometry along each fiber,*
- *the fibered flow \widehat{P}^* over $i(\Lambda \setminus \text{Sing}(X))$ is conjugated by I near the zero-section to the rescaled sectional Poincaré flow P^* :*

$$\widehat{P}^* = I \circ P^* \circ I^{-1}.$$

The linear Poincaré flow introduced by Liao [Li₁] was compactified by Li, Gan and Wen [LGW], who called it *extended linear Poincaré flow*. Liao also introduced its rescaling [Li₂]. Gan and Yang [GY] considered the rescaled sectional Poincaré flow and proved some uniform properties.

Identifications structures for fibered flows. Since P^* is defined as a sectional flow over φ , the holonomy by the flow gives a projection between fibers of points close. The fibered flow thus comes with an additional structure, that we call *C^k -identification*: let U be an open set in $\Lambda \setminus \text{Sing}(X)$, for any points $x, y \in U$ close enough, there is a C^k -diffeomorphism $\pi_{y,x} : \mathcal{N}_y \rightarrow \mathcal{N}_x$ which satisfies $\pi_{z,x} \circ \pi_{y,z} = \pi_{y,x}$. These identifications $\pi_{x,y}$ satisfy several properties (called *compatibility with the flow*), such as some invariance. See Section 3.2 for precise definitions.

¹When x is periodic and t is the period of x , this map is defined by Poincaré to study the dynamics in a neighborhood of a regular periodic orbit.

1.4 Generalization of Mañé-Pujals-Sambarino's theorem for flows

Let us consider an invariant compact set Λ for a C^2 flow φ such that the linear Poincaré flow on $\Lambda \setminus \text{Sing}(X)$ admits a dominated splitting $\mathcal{N} = \mathcal{E} \oplus \mathcal{F}$. Under some assumptions, Theorem A' asserts that the tangent flow is then dominated. The existence of a dominated splitting $TM|_\Lambda = E \oplus F$ with $\dim(E) = 1$ and $X \subset F$ is equivalent to the fact that \mathcal{E} is uniformly contracted by the rescaled linear Poincaré flow (see Proposition 7.3). It is thus reduced to prove that the one-dimensional bundle \mathcal{E} of the splitting of the two-dimensional bundle \mathcal{N} is uniformly contracted by the extended sectional Poincaré flow P^* .

For C^2 surface diffeomorphisms, the existence of a dominated splitting implies that the (one-dimensional) bundles are uniformly hyperbolic, under mild assumptions: this is one of the main results of Pujals and Sambarino [PS₁]. A result implying the hyperbolicity for one-dimensional endomorphisms was proved before by Mañé [M₁].

Our main technical theorem is to extend that technique to the case of local fibered flows with 2-dimensional dominated fibers. As introduced in section 1.3 we will assume the existence of identifications compatible with the flow, over an open set U . We will assume that, on a neighborhood of the complement $\Lambda \setminus U$, the fibered flow contracts the bundle \mathcal{E} : this is a non-symmetric assumption on the splitting $\mathcal{E} \oplus \mathcal{F}$. See Section 3 for the precise definitions.

Theorem C (Hyperbolicity of one-dimensional extremal bundle). *Consider a C^2 local fibered flow (N, P) over a topological flow (K, φ) on a compact metric space such that:*

1. *there is a dominated splitting $\mathcal{N} = \mathcal{E} \oplus \mathcal{F}$ and \mathcal{E}, \mathcal{F} have one-dimensional fibers,*
2. *there exists a C^2 -identification compatible with (P_t) on an open set U ,*
3. *\mathcal{E} is uniformly contracted on an open set V containing $K \setminus U$.*

Then, one of the following properties occurs:

- *there exists a periodic orbit $\mathcal{O} \subset K$ such that $\mathcal{E}|_{\mathcal{O}}$ is not uniformly contracted,*
- *there exists a normally expanded irrational torus,*
- *\mathcal{E} is uniformly contracted above K .*

This theorem is based on the works initiated by Mañé [M₁] and Pujals-Sambarino [PS₁], but we have to address additional difficulties:

- The time of the dynamical system is not discrete. This produces some shear between pieces of orbits that remain close. In the non-singular case, Arroyo and Rodriguez-Hertz [ARH] already met that difficulty.
- Pujals-Sambarino's theorem does not hold in general for fibered systems. In our setting, the existence of an identification structure is essential.
- We adapt the notion of “induced hyperbolic returns” from [CP]: this allows us to work with the induced dynamics on U where the identifications are defined.
- In the setting of local flows, we have to replace some global arguments in [PS₁, ARH].
- The role of the two bundles \mathcal{E} and \mathcal{F} is non-symmetric. In particular we do not have the topological hyperbolicity of \mathcal{F} . The construction of Markovian boxes (Section 5) then requires other ideas, which can be compared to arguments in [CPS].

Structure of the paper

In Section 2, we compactify the rescaled sectional Poincaré flow and prove Theorem B. Local fibered flows are studied systematically in Section 3. The proof of Theorem C occupies Sections 4 to 6. The Theorem A' is obtained in Section 7. The proofs of global genericity results, including the Main Theorem, Corollary 1.1 and Theorem A are completed in Section 8.

Acknowledgements. We are grateful to S. Gan, R. Potrie, E. Pujals and L. Wen for discussions related to this work. We also thank the Université Paris 11, Soochow University and Pekin University for their hospitality.

2 Compactification of the sectional flow

In this section we do not restrict $\dim(M)$ to be equal to 3 and we prove Theorem B (see Theorem 2.7). Let X be a C^k vector field, for some $k \geq 1$, and let $(\varphi_t)_{t \in \mathbb{R}}$ be its associated flow. We also assume that $DX(\sigma)$ is invertible at each singularity. In particular $\text{Sing}(X)$ is finite.

Several flows associated to φ have already been used in [Li₂, LGW, GY]. We describe here slightly different constructions and introduce the “extended rescaled sectional Poincaré flow”.

2.1 Linear flows

We associate to $(\varphi_t)_{t \in \mathbb{R}}$ several C^{k-1} linear and projective flows.

The tangent flow $(D\varphi_t)_{t \in \mathbb{R}}$ is the flow on the tangent bundle TM which fibers over $(\varphi_t)_{t \in \mathbb{R}}$ and is obtained by differentiation.

The unit tangent flow $(U\varphi_t)_{t \in \mathbb{R}}$ is the flow on the unit tangent bundle T^1M obtained from $(D\varphi_t)_{t \in \mathbb{R}}$ by normalization:

$$U\varphi_t.v = \frac{D\varphi_t.v}{\|D\varphi_t.v\|} \text{ for } v \in T^1M.$$

Sometimes we prefer to work with the projective bundle PTM . The unit tangent flow induces a flow on this bundle, that we also denote by $(U\varphi_t)_{t \in \mathbb{R}}$ for simplicity.

The normal flow $(\mathcal{N}\varphi_t)_{t \in \mathbb{R}}$. For each $(x, u) \in T^1M$ we denote $\mathcal{N}T_x^1M$ as the vector subspace orthogonal to $\mathbb{R}.u$ in T_xM . This defines a vector bundle $\mathcal{N}T^1M$ over the compact manifold T^1M . We define the normal flow $(\mathcal{N}\varphi_t)_{t \in \mathbb{R}}$ on $\mathcal{N}T^1M$ which fibers above the unit tangent flow as the orthogonal projection $\mathcal{N}\varphi_t.v$ of $D\varphi_t.v$ on $(D\varphi_t.u)^\perp$.

The linear Poincaré flow $(\psi_t)_{t \in \mathbb{R}}$. The normal bundle $\mathcal{N}(M \setminus \text{Sing}(X))$ over the space of non-singular points x is the union of the vector subspaces $\mathcal{N}_x = X(x)^\perp$. It can be identified with the restriction of the bundle $\mathcal{N}T^1M$ over the space of pairs $(x, \frac{X(x)}{\|X(x)\|})$ for $x \in M \setminus \text{Sing}(X)$. The linear Poincaré flow $(\psi_t)_{t \in \mathbb{R}}$ is the restriction of $(\mathcal{N}\varphi_t)$ to $\mathcal{N}(M \setminus \text{Sing}(X))$.

2.2 Lifted and sectional flows

The sectional Poincaré flow $(P_t)_{t \in \mathbb{R}}$. There exists $r_0 > 0$ such that the ball $B(0, r_0)$ in each fiber of the bundle $\mathcal{N}(M \setminus \text{Sing}(X))$ projects on M diffeomorphically by the exponential map. For each $x \in M \setminus \text{Sing}(X)$, there exists $r_x \in (0, r_0)$ such that for any $t \in [0, 1]$, the holonomy map of the flow induces a local diffeomorphism P_t from $B(0_x, r_x) \subset \mathcal{N}_x$ to a neighborhood of

$0_{\varphi_t(x)}$ in $B(0_{\varphi_t(x)}, r_0) \subset \mathcal{N}_{\varphi_t(x)}$. This extends to a local flow $(P_t)_{t \in \mathbb{R}}$ in a neighborhood of the 0-section in $\mathcal{N}(M \setminus \text{Sing}(X))$, that is called the sectional Poincaré flow. It is tangent to $(\psi_t)_{t \in \mathbb{R}}$ at the 0-section of $\mathcal{N}(M \setminus \text{Sing}(X))$.

The normal bundle and the sectional Poincaré flow are C^k .

The *lifted flow* $(\mathcal{L}\varphi_t)_{t \in \mathbb{R}}$. Similarly, for each $t \in [0, 1]$ and $x \in M$, the map

$$\mathcal{L}\varphi_t: y \mapsto \exp_{\varphi_t(x)}^{-1} \circ \varphi_t \circ \exp_x(y)$$

sends diffeomorphically a neighborhood of 0 in $T_x M$ to a neighborhood of 0 in $B(0, r_0) \subset T_{\varphi_t(x)} M$. This extends to a local flow $(\mathcal{L}\varphi_t)_{t \in \mathbb{R}}$ in a neighborhood of the 0-section of TM , that is called the lifted flow. It is tangent to $(D\varphi_t)_{t \in \mathbb{R}}$ at the 0-section.

The *fiber-preserving lifted flow* $(\mathcal{L}_0\varphi_t)_{t \in \mathbb{R}}$. We can choose not to move the base point x and obtain a fiber-preserving flow $(\mathcal{L}_0\varphi_t)_{t \in \mathbb{R}}$, defined by:

$$\mathcal{L}_0\varphi_t(y) = \exp_x^{-1} \circ \varphi_t \circ \exp_x(y).$$

Since the 0-section is not preserved, this is not a local flow and it will be considered only for short times.

2.3 Rescaled flows

The *rescaled sectional and linear Poincaré flows* $(P_t^*)_{t \in \mathbb{R}}$, $(\psi_t^*)_{t \in \mathbb{R}}$. Since $DX(\sigma)$ is invertible at each singularity, there exists $\beta > 0$ such that at any $x \in M \setminus \text{Sing}(X)$

$$r_x > \beta \|X(x)\|.$$

We can thus rescale the sectional Poincaré flow. We get for each $x \in M \setminus \text{Sing}(X)$ and $t \in [0, 1]$ a map P_t^* which sends diffeomorphically $B(0, \beta) \subset \mathcal{N}_x$ to $\mathcal{N}_{\varphi_t(x)}$, defined by:

$$P_t^*(y) = \|X(\varphi_t(x))\|^{-1} \cdot P_t(\|X(x)\| \cdot y).$$

Again, this induces a local flow $(P_t^*)_{t \in \mathbb{R}}$ in a neighborhood of the 0-section in $\mathcal{N}(M \setminus \text{Sing}(X))$, that is called the *rescaled sectional Poincaré flow*. Its tangent map at the 0-section defines the rescaled linear Poincaré flow (ψ_t^*) .

The *rescaled lifted flow* $(\mathcal{L}\varphi_t^*)_{t \in \mathbb{R}}$ **and the *rescaled tangent flow*** $(D\varphi_t^*)_{t \in \mathbb{R}}$. The rescaled lifted flow is defined on a neighborhood of the 0-section in TM by

$$\mathcal{L}\varphi_t^*(y) = \|X(\varphi_t(x))\|^{-1} \cdot \mathcal{L}\varphi_t(\|X(x)\| \cdot y).$$

Its tangent map at the 0-section defines the rescaled tangent flow $(D\varphi_t^*)_{t \in \mathbb{R}}$.

The *rescaled fiber-preserving lifted flow* $(\mathcal{L}_0\varphi_t^*)_{t \in \mathbb{R}}$ is defined similarly:

$$\mathcal{L}_0\varphi_t^*(y) = \|X(x)\|^{-1} \cdot \mathcal{L}_0\varphi_t(\|X(x)\| \cdot y).$$

2.4 Blowup

We will consider a compactification of $M \setminus \text{Sing}(X)$ and of the tangent bundle $TM|_{M \setminus \text{Sing}(X)}$ which allows to extend the line field $\mathbb{R}X$. This is given by the classical blowup.

The manifold \widehat{M} . We can blowup M at each singularity of X and get a new compact manifold \widehat{M} and a projection $p: \widehat{M} \rightarrow M$ which is one-to-one above $M \setminus \text{Sing}(X)$. Each singularity $\sigma \in \text{Sing}(X)$ has been replaced by the projectivization $PT_\sigma M$.

More precisely, at each (isolated) singularity σ , one can add $T_\sigma^1 M$ to $M \setminus \{\sigma\}$ in order to build a manifold with boundary. Locally, it is defined by the chart $[0, \varepsilon) \times T_\sigma^1 M \rightarrow (M \setminus \{\sigma\}) \cup T_\sigma^1 M$ given by:

$$(s, u) \mapsto \begin{cases} \exp(s.u) & \text{if } s \neq 0, \\ u & \text{if } s = 0. \end{cases}$$

One then gets \widehat{M} by identifying points $(0, u)$ and $(0, -u)$ on the boundary.

It is sometimes convenient to lift the dynamics on $M \setminus \{\sigma\}$ near σ and work in the local coordinates $(-\varepsilon, \varepsilon) \times T_\sigma^1 M$. These coordinates define a double covering of an open subset of the blowup \widehat{M} and induce a chart from the quotient $(-\varepsilon, \varepsilon) \times T_\sigma^1 M / (s, u) \sim (-s, -u)$ to a neighborhood of $p^{-1}(\sigma)$ in \widehat{M} .

The extended flow $(\widehat{\varphi}_t)_{t \in \mathbb{R}}$. The following result is proved in [T, section 3].

Proposition 2.1. *The flow $(\varphi_t)_{t \in \mathbb{R}}$ induces a C^{k-1} flow $(\widehat{\varphi}_t)_{t \in \mathbb{R}}$ on \widehat{M} which is associated to a C^{k-1} vector field \widehat{X} . For $\sigma \in \text{Sing}(X)$, this flow preserves $PT_\sigma M$, and acts on it as the projectivization of $D\varphi_t(\sigma)$; the vector field \widehat{X} coincides at $u \in PT_\sigma M$ with $DX(\sigma).u$ in $T_u(PT_\sigma M) \cong T_\sigma M / \mathbb{R}.u$.*

In particular the tangent bundle $T\widehat{M}$ extends $TM|_{M \setminus \text{Sing}(X)}$, the linear flow $D\widehat{\varphi}$ extend $D\varphi$ and the vector field \widehat{X} extends X . Note that each eigendirection u of $DX(\sigma)$ at a singularity σ induces a singularity of \widehat{X} .

Remark. In [T], the vector field and the flow are extended locally on the space $(-\varepsilon, \varepsilon) \times T_\sigma^1 M$, but the proof shows that these extensions are invariant under the map $(s, u) \mapsto (-s, -u)$, hence are also defined on \widehat{M} .

The extended bundle \widehat{TM} and extended tangent flow $(\widehat{D\varphi}_t)_{t \in \mathbb{R}}$. One associates to \widehat{M} the bundle \widehat{TM} which is the pull-back of the bundle $\pi: TM \rightarrow M$ over M by the map $p: \widehat{M} \rightarrow M$. It can be obtained as the restriction of the first projection $\widehat{M} \times TM \rightarrow \widehat{M}$ to the set of pairs (x, v) such that $p(x) = \pi(v)$. It is naturally endowed with the pull back metric of TM and it is trivial in a neighborhood of preimages $p^{-1}(z)$, $z \in M$.

The tangent flow $(D\varphi_t)_{t \in \mathbb{R}}$ can be pull back to \widehat{TM} as a C^{k-1} linear flow $(\widehat{D\varphi}_t)_{t \in \mathbb{R}}$ that we call extended tangent flow.

The extended line field $\widehat{\mathbb{R}X}$. The vector field X induces a line field $\mathbb{R}X$ on $M \setminus \text{Sing}(X)$ which admits an extension to \widehat{TM} . It is defined locally as follows.

Proposition 2.2. *At each singularity σ , let U be a small neighborhood in M and $\widehat{U} = (U \setminus \{\sigma\}) \cup PT_\sigma M$ be a neighborhood of $PT_\sigma M$. Then, the map $x \mapsto \frac{\exp_\sigma^{-1}(x)}{\|X(x)\|}$ on $U \setminus \sigma$ extends to \widehat{U} as a C^{k-1} -map which coincides at $u \in PT_\sigma M$ with $\frac{u}{\|DX(\sigma).u\|}$, and the map $x \mapsto \frac{\|X(x)\|}{d(x, \sigma)}$ on $U \setminus \{\sigma\}$ extends to \widehat{U} as a C^{k-1} -map which coincides at $u \in PT_\sigma M$ with $\|DX(\sigma).u\|$.*

In the local coordinates $(-\varepsilon, \varepsilon) \times T_\sigma^1 M$ associated to $\sigma \in \text{Sing}(X)$, the lift of the vector field $X_1 := X/\|X\|$ on $M \setminus \text{Sing}(X)$ extends as a (non-vanishing) C^{k-1} section $\widehat{X}_1: (-\varepsilon, \varepsilon) \times T_\sigma^1 M \rightarrow \widehat{TM}$. For each $x = (0, u) \in p^{-1}(\sigma)$, one has

$$\widehat{X}_1(x) = \frac{DX(\sigma).u}{\|DX(\sigma).u\|}.$$

A priori, the extension of X_1 is not preserved by the symmetry $(-s, -u) \sim (s, u)$ and is not defined in \widehat{TM} . However, the line field $\widehat{\mathbb{R}X}_1$ is invariant by the local symmetry $(s, u) \mapsto (-s, -u)$, hence induces a C^{k-1} -line field $\widehat{\mathbb{R}X}$ on \widehat{TM} invariant by $(\widehat{D\varphi}_t)_{t \in \mathbb{R}}$.

Proof. In a local chart near a singularity, we have

$$X(x) = \int_0^1 DX(r.x).x \, dr.$$

Working in the local coordinates $(s, u) \in (-\varepsilon, \varepsilon) \times T_\sigma^1 M$, we get

$$X(x) = \int_0^1 DX(rs.u) \, dr \cdot s.u.$$

This allows us to define a C^{k-1} section in a neighborhood of $p^{-1}(\sigma)$ defined by

$$\bar{X}: (s, u) \mapsto \int_0^1 DX(rs.u) \, dr.u.$$

This section is C^{k-1} , is parallel to X (when $s \neq 0$) and does not vanish. Consequently $\frac{\bar{X}}{\|\bar{X}\|}$ is C^{k-1} and extends the vector field $X_1 := X/\|X\|$ as required.

Since \bar{X} extends as $DX(\sigma).u$ at $u \in PT_\sigma M$, then X_1 extends as $DX(\sigma).u/\|DX(\sigma).u\|$.

Note also that for $s \neq 0$, $\|\bar{X}(s.u)\|$ coincides with $\|X(x)\|/d(x, \sigma)$ where $su = x$ is a point of $M \setminus \{\sigma\}$ close to σ . Since \bar{X} is C^{k-1} and does not vanish, $(s, u) \mapsto \|\bar{X}(s.u)\|$ extends as a C^{k-1} -function in the local coordinates $(-\varepsilon, \varepsilon) \times T_\sigma^1 M$. It is invariant by the symmetry $(s, u) \sim (-s, -u)$, hence the maps $x \rightarrow \frac{\exp_\sigma^{-1}(x)}{\|X(x)\|}$ and $x \mapsto \|X(x)\|/d(x, \sigma)$ for $x \in M \setminus \{\sigma\}$ close to σ extends as a C^{k-1} on a neighborhood of $PT_\sigma M$ in \widehat{M} . \square

The extended normal bundle $\widehat{\mathcal{N}M}$ and extended linear Poincaré flow $(\widehat{\psi}_t)_{t \in \mathbb{R}}$. The orthogonal spaces to the lines of $\widehat{\mathbb{R}X}$ define a C^{k-1} linear bundle $\widehat{\mathcal{N}M}$. Since $\widehat{\mathbb{R}X}$ is preserved by the extended tangent flow, the projection of $(\widehat{D\varphi}_t)_{t \in \mathbb{R}}$ defines the C^{k-1} extended linear Poincaré flow $(\widehat{\psi}_t)_{t \in \mathbb{R}}$ on $\widehat{\mathcal{N}M}$.

Alternative construction. One can also embed $M \setminus \text{Sing}(X)$ in PTM by the map $x \mapsto (x, \frac{X(x)}{\|X(x)\|})$, and take the closure. This set is invariant by the unit flow. This compactification depends on the vector field X and not only on the finite set $\text{Sing}(X)$. It is sometimes called *Nash blowup*, see [No].

When $DX(\sigma)$ is invertible at each singularity, Proposition 2.2 shows that the closure is homeomorphic to \widehat{M} . The restriction of the normal bundle $\mathcal{N}T^1 M$ to the closure of $M \setminus \text{Sing}(X)$ in PTM gives the normal bundle $\widehat{\mathcal{N}M}$. This is the approach followed in [LGW] in order to compactify of the linear Poincaré flow.

2.5 Compactifications of non-linear local fibered flows

The rescaled flows introduced above extend to the bundles \widehat{TM} or $\widehat{\mathcal{N}M}$. In the following, one will assume that $DX(\sigma)$ is invertible at each singularity and (without loss of generality) that the metric on M is flat near each singularity of X .

Related to the “local C^k -fibered flow” in Definition 1.2, we will also use the following notion.

Definition 2.3. Consider a continuous Riemannian vector bundle \mathcal{N} over a compact metric space K . A map $H: \mathcal{N} \rightarrow \mathcal{N}$ is C^k -fibered, if it fibers over a homeomorphism h of K and if each induced map $H_x: \mathcal{N}_x \rightarrow \mathcal{N}_{h(x)}$ is C^k and depends continuously on x for the C^k -topology.

The extended lifted flow. The following proposition compactifies the rescaled lifted flow $(\mathcal{L}\varphi_t^*)_{t \in \mathbb{R}}$ (and the rescaled tangent flow $(D\varphi_t^*)_{t \in \mathbb{R}}$) as local fibered flows on \widehat{TM} .

Proposition 2.4. *The rescaled lifted flow $(\mathcal{L}\varphi_t^*)_{t \in \mathbb{R}}$ extends as a local C^k -fibered flow on \widehat{TM} . The rescaled tangent flow $(D\varphi_t^*)_{t \in \mathbb{R}}$ extends as a linear flow.*

Moreover, there exists $\beta > 0$ such that, for each $t \in [0, 1]$, $\sigma \in \text{Sing}(X)$ and $x = u \in p^{-1}(\sigma)$, on the ball $B(0_x, \beta) \subset \widehat{T_x M}$ the map $\mathcal{L}\varphi_t^$ writes as:*

$$y \mapsto \frac{\|DX(\sigma).u\|}{\|DX(\sigma) \circ D\varphi_t(\sigma).u\|} D\varphi_t(\sigma).y. \quad (1)$$

Before proving the proposition, one first shows:

Lemma 2.5. *The function $(x, t) \mapsto \frac{\|X(x)\|}{\|X(\varphi_t(x))\|}$ on $(M \setminus \text{Sing}(X)) \times \mathbb{R}$ extends as a positive C^{k-1} function $\widehat{M} \times \mathbb{R} \rightarrow \mathbb{R}_+$ which is equal to $\frac{\|DX(\sigma).u\|}{\|DX(\sigma) \circ D\varphi_t(\sigma).u\|}$ when $x = u \in p^{-1}(\sigma)$.*

The map from $TM|_{M \setminus \text{Sing}(X)}$ into itself which sends $y \in T_x M$ to $\|X(x)\|.y$, extends as a continuous map of \widehat{TM} which vanishes on the set $p^{-1}(\text{Sing}(X))$ and is C^∞ -fibered.

Proof. From Proposition 2.2, in the local chart of $0 = \sigma \in \text{Sing}(X)$, the map $x \mapsto \frac{\|X(x)\|}{\|x\|}$ extends as a C^{k-1} function which coincides at $u \in PT_\sigma M$ with $\|DX(\sigma).u\|$ and does not vanish. We also extend the map $(x, t) \mapsto \|\varphi_t(x)\|/\|x\|$ as a C^{k-1} map on $\widehat{M} \times \mathbb{R}$ which coincides with $\|D\varphi_t(\sigma).u\|$ when $x = u$. The proof is similar to the proof of Proposition 2.2. This implies the first part of the lemma.

For the second part, one considers the product of the C^{k-1} function $x \mapsto \frac{\|X(x)\|}{\|x\|}$ with the C^∞ -fibered map which extends $y \mapsto \|x\|.y$. \square

Proof of Proposition 2.4. In local coordinates, the local flow $(\mathcal{L}\varphi_t^*)_{t \in \mathbb{R}}$ in $T_x M$ acts like:

$$\begin{aligned} \mathcal{L}\varphi_t^*(y) &= \|X(\varphi_t(x))\|^{-1} (\varphi_t(x + \|X(x)\|.y) - \varphi_t(x)) \\ &= \frac{\|X(x)\|}{\|X(\varphi_t(x))\|} \int_0^1 D\varphi_t(x + r\|X(x)\|.y) . y \, dr. \end{aligned} \quad (2)$$

By Lemma 2.5, $\frac{\|X(x)\|}{\|X(\varphi_t(x))\|}$ and $\|X(x)\|.y$ extend as a continuous map and as a C^∞ -fibered homeomorphism respectively; hence $(\mathcal{L}\varphi_t^*)_{t \in \mathbb{R}}$ extends continuously at $x = (0, u) \in p^{-1}(\sigma)$ as in (1). The extended flow is C^k along each fiber. Moreover, (2) implies that it is C^{k-1} -fibered. For $x \in M \setminus \text{Sing}(X)$, the k^{th} derivative along the fibers is equal to

$$\frac{\|X(x)\|^k}{\|X(\varphi_t(x))\|} D^k \varphi_t(x + \|X(x)\|.y).$$

This converges to $\frac{\|DX(\sigma).u\|}{\|DX(\sigma) \circ D\varphi_t(\sigma).u\|} D\varphi_t(\sigma)$ when $k = 1$ and to 0 for $k > 1$. The extended rescaled lifted flow is thus a local C^k -fibered flow defined on a uniform neighborhood of the 0-section.

From Lemma 2.5, the rescaled linear flow $(D\varphi_t^*)_{t \in \mathbb{R}}$ extends to \widehat{TM} and coincides at $x = u \in \pi^{-1}(\sigma)$ with the map defined by (1). From (2), it coincides also with the flow tangent to $(\mathcal{L}\varphi_t^*)_{t \in \mathbb{R}}$ at the 0-section.

In order to define $\mathcal{L}\varphi_t^*$ on the whole bundle \widehat{TM} (and get a fibered flow as in Definition 1.2), one first glues each diffeomorphism $\mathcal{L}\varphi_t^*$ for $t \in [0, 1]$ on a small uniform neighborhood of 0 with the linear map $D\varphi_t^*$ outside a neighborhood of 0 in such a way that $\mathcal{L}\varphi_0^* = \text{Id}$. One then defines $\mathcal{L}\varphi_t^*$ for other times by:

$$\mathcal{L}\varphi_{-t}^* = (\mathcal{L}\varphi_t^*)^{-1} \text{ for } t > 0,$$

$$\mathcal{L}\varphi_{n+t}^* = \mathcal{L}\varphi_t^* \circ \mathcal{L}\varphi_1^* \circ \cdots \circ \mathcal{L}\varphi_1^* \text{ (} n+1 \text{ terms), for } t \in [0, 1] \text{ and } n \in \mathbb{N}.$$

□

In a same way we compactify the rescaled fiber-preserving lifted flow $(\mathcal{L}_0\varphi_t^*)$.

Proposition 2.6. *The rescaled fiber-preserving lifted flow $(\mathcal{L}_0\varphi_t^*)_{t \in \mathbb{R}}$ extends as a local C^k -fibered flow on \widehat{TM} . More precisely, for each $x \in \widehat{TM}$, it defines a C^k -map $(t, y) \mapsto \mathcal{L}_0\varphi_t^*(y)$ from $\mathbb{R} \times \widehat{T_x M}$ to $\widehat{T_x M}$ which depends continuously on x for the C^k -topology.*

Moreover there exists $\beta > 0$ such that, for each $t \in [0, 1]$, $\sigma \in \text{Sing}(X)$ and $x = u \in p^{-1}(\sigma)$, on the ball $B(0, \beta) \subset \widehat{T_x M}$ the map $\mathcal{L}_0\varphi_t^$ has the form:*

$$y \mapsto D\varphi_t(\sigma).y + \frac{D\varphi_t(\sigma).u - u}{\|DX(\sigma).u\|}. \quad (3)$$

Proof. In the local coordinates the flow acts on $B(0, \beta) \subset T_x M$ as:

$$\mathcal{L}_0\varphi_t^*(y) = \|X(x)\|^{-1} (\varphi_t(x + \|X(x)\|.y) - x) \quad (4)$$

$$= \int_0^1 D\varphi_t(x + r\|X(x)\|.y).y \, dr + \frac{\varphi_t(x) - x}{\|X(x)\|}. \quad (5)$$

Arguing as in Proposition 2.2 and Lemma 2.5, for each t , the map $x \mapsto \frac{\varphi_t(x) - x}{\|X(x)\|}$ with $x \neq \sigma$ close to σ extends for the C^0 -topology by $\|DX(\sigma).u\|^{-1}(D\varphi_t(\sigma).u - u)$ at points $(0, u) \in PT_\sigma M$. Since X is C^k , these maps are all C^k and depends continuously with x for the C^k -topology.

As before, the integral $\int_0^1 D\varphi_t(x + r\|X(x)\|.y).y \, dr$ extends as $D\varphi_t(\sigma).y$ at $p^{-1}(\sigma)$. For each x , the map $(t, y) \mapsto \int_0^1 D\varphi_t(x + r\|X(x)\|.y).y \, dr$ is C^k (this is checked on the formulas, considering separately the cases $x \in M \setminus \text{Sing}(X)$ and $x \in p^{-1}(\text{Sing}(X))$). Since X is C^k , this map depends continuously on x for the C^{k-1} -topology. The k^{th} derivative with respect to y is continuous in x , for the same reason as in the proof of Proposition 2.4. For $x \in M \setminus \text{Sing}(X)$, the derivative with respect to t of the map above is $(t, y) \mapsto \int_0^1 DX(\varphi_t(x + r\|X(x)\|.y)).y \, dr$ and it converges as $x \rightarrow \sigma$ towards $(x, y) \mapsto DX(\varphi_t(\sigma)).y$ for the C^{k-1} -topology (again using that X is C^k). Hence the first term of (5) is a C^k -function of (t, y) which depends continuously on x for the C^k -topology.

As in Proposition 2.4, this proves that $(\mathcal{L}_0\varphi_t^*)_{t \in \mathbb{R}}$ extends as a local C^k -fibered flow having the announced properties. □

The extended rescaled sectional Poincaré flow. We also obtain a compactification of the rescaled sectional Poincaré flow $(P_t^*)_{t \in \mathbb{R}}$ (and of the rescaled linear Poincaré flow $(\psi_t^*)_{t \in \mathbb{R}}$). This implies Theorem B.

Theorem 2.7. *If X is C^k , $k \geq 1$, and if $DX(\sigma)$ is invertible at each singularity, the rescaled sectional Poincaré flow $(P_t^*)_{t \in \mathbb{R}}$ extends as a C^k -fibered flow on a neighborhood of the 0-section in $\widehat{\mathcal{N}M}$. Moreover, there exists $\beta' > 0$ such that for each $t \in [0, 1]$, $\sigma \in \text{Sing}(X)$ and $x = u \in p^{-1}(\sigma)$, on the ball $B(0, \beta') \subset \widehat{\mathcal{N}_x M}$ the map P_t^* writes as:*

$$y \mapsto \frac{\|DX(\sigma).u\|}{\|DX(\sigma) \circ D\varphi_t(\sigma).u\|} D\varphi_{t+\tau}(\sigma).y + \frac{D\varphi_{t+\tau}(\sigma).u - D\varphi_t(\sigma).u}{\|DX(\sigma) \circ D\varphi_t(\sigma).u\|},$$

where τ is a C^k function of $(t, y) \in [0, 1] \times B(0, \beta')$ which depends continuously on x for the C^k -topology, such that $\tau(x, t, 0) = 0$.

As a consequence, the rescaled linear Poincaré flow (ψ_t^*) extends as a continuous linear flow: for each $x \in \widehat{M}$, each $t \in \mathbb{R}$ and each $v \in \widehat{\mathcal{N}_x M}$ the image $\psi_t^*.v$ coincides with the normal projection of $D\varphi_t^*.v \in \widehat{T_{\widehat{\varphi}_t(x)} M}$ on $\widehat{\mathcal{N}_{\widehat{\varphi}_t(x)} M}$.

Proof. For each singularity σ , we work with the local coordinates $(-\varepsilon, \varepsilon) \times T_\sigma^1 M$ and prove that the rescaled sectional Poincaré flow extends as a local C^k -fibered flow. Since the rescaled sectional Poincaré flow is invariant by the symmetry $(s, u) \mapsto (-s, -u)$, this implies the result above a neighborhood of $p^{-1}(\sigma)$ in \widehat{M} , and hence above the whole manifold \widehat{M} .

The image of $y \in B(0, \beta) \subset \mathcal{N}_x$ by the rescaled sectional Poincaré flow is the unique point of the curve in $T_{\varphi_t(x)} M$

$$\tau \mapsto \mathcal{L}_0 \varphi_\tau^* \circ \mathcal{L} \varphi_t^*(y)$$

which belongs to $\mathcal{N}_{\varphi_t(x)}$. In the local coordinates $x = (s, u) \in (-\varepsilon, \varepsilon) \times T_\sigma^1 M$ near σ , it corresponds to the unique value $\tau = \tau(x, t, y)$ such that the following function vanishes:

$$\Theta(x, t, y, \tau) = \left\langle \mathcal{L}_0 \varphi_\tau^* \circ \mathcal{L} \varphi_t^*(y), \frac{X(\varphi_t(x))}{\|X(\varphi_t(x))\|} \right\rangle.$$

From the previous propositions the map $(y, \tau) \mapsto \Theta(x, t, y, \tau)$ is C^k and depends continuously on (x, t) for the C^k -topology and is defined at any $x \in \widehat{M}$. When $x = u \in p^{-1}(\sigma)$, the first part of the scalar product in Θ is given by the two propositions above. According to Proposition 2.2 the second part $\widehat{X}_1(x)$ becomes equal to

$$\frac{DX(\sigma) \circ D\varphi_t(\sigma).u}{\|DX(\sigma) \circ D\varphi_t(\sigma).u\|}.$$

Claim. For any $t \in [0, 1]$ and for $x \in p^{-1}(\sigma)$, the derivative $\frac{\partial \Theta}{\partial \tau}|_{\tau=0}(x, t, 0)$ is non-zero.

Proof. Indeed, by the previous proposition this is equivalent to

$$\left\langle \frac{\partial}{\partial \tau}|_{\tau=0}(D\varphi_\tau(\sigma).v), \frac{DX(\sigma) \circ D\varphi_t(\sigma).u}{\|DX(\sigma) \circ D\varphi_t(\sigma).u\|} \right\rangle \neq 0,$$

where $v = \widehat{\varphi}_t(u) = D\varphi_t(\sigma).u / \|D\varphi_t(\sigma).u\|$. Thus the condition becomes

$$\frac{\|DX(\sigma) \circ D\varphi_t(\sigma).u\|^2}{\|DX(\sigma) \circ D\varphi_t(\sigma).u\|} \neq 0,$$

which is satisfied. \square

By the implicit function theorem and compactness, there exists $\beta' > 0$ and, for each x , a C^k map $(y, t) \mapsto \tau(x, t, y)$ which depends continuously on x for the C^k -topology such that

$$\Theta(x, t, y, \tau(x, t, y)) = 0,$$

for each $x \in \widehat{M}$ close to $PT_\sigma M$, each $t \in [0, 1]$ and each $y \in B(0, \beta')$. The rescaled sectional Poincaré flow is thus locally given by the composition:

$$(x, t, y) \mapsto \mathcal{L}_0 \varphi_{\tau(x, t, y)}^* \circ \mathcal{L} \varphi_t^*(y), \quad (6)$$

which extends as a C^k -fibered flow. The formula at $x = u \in p^{-1}(\sigma)$ is obtained from the expressions in the previous propositions.

We now compute the rescaled linear Poincaré flow as the tangent map to the rescaled sectional Poincaré flow along the 0-section. We fix $x \in \widehat{M}$ and its image $x' = \widehat{\varphi}_t(x)$. We take $y \in \mathcal{N}_x$ and its image $y' \in \widehat{T_{\widehat{\varphi}_t(x)} M}$ by $\mathcal{L} \varphi_t^*$. Working in the local coordinates $(-\varepsilon, \varepsilon) \times T_\sigma^1 M$ and using Proposition 2.2 and formulas (3) and (4) we get

$$\frac{\partial}{\partial \tau}|_{\tau=0} \mathcal{L}_0 \varphi_\tau^*(0_{x'}) = \widehat{X}_1(x').$$

Note that $\tau(x', t, 0) = 0$, we have:

$$\begin{aligned} DP_t^*(0) &= \left(\frac{\partial}{\partial y'} \Big|_{y'=0} \mathcal{L}_0 \varphi_0^*(y') \right) \circ \left(\frac{\partial}{\partial y} \Big|_{y=0} \mathcal{L}_0 \varphi_t^*(y) \right) + \frac{\partial \tau}{\partial y} \Big|_{y=0} \cdot \left(\frac{\partial}{\partial \tau} \Big|_{\tau=0} \mathcal{L}_0 \varphi_\tau^*(0_{x'}) \right) \\ &= D\varphi_t^*(x) + \frac{\partial \tau}{\partial y} \Big|_{y=0} \cdot \widehat{X}_1(x'). \end{aligned}$$

On the other hand from (6) and the definitions of Θ, τ we have

$$\langle DP_t^*(0), \widehat{X}_1(x') \rangle = 0,$$

hence $DP_t^*(0)$ coincides with the normal projection of $D\varphi_t^*(x)$ on the linear sub-space of $\widehat{T_{x'}M}$ orthogonal to $\widehat{X}_1(x')$, which is $\widehat{\mathcal{N}_{x'}M}$. \square

Proof of Theorem B. Let Λ be the compact invariant set in the assumption. Recall the blowup \widehat{M} and the projection $p : \widehat{M} \rightarrow M$. We define $\widehat{\Lambda}$ as the closure of $p^{-1}(\Lambda \setminus \text{Sing}(X))$ in \widehat{M} . Since the flow φ on M induces a flow $\widehat{\varphi}$ on \widehat{M} , we know that the restriction of φ to $\Lambda \setminus \text{Sing}(X)$ embeds in $(\widehat{\Lambda}, \widehat{\varphi})$ through the map $i = p^{-1}$.

The metric on the bundle \widehat{TM} over \widehat{M} is the pull back metric of TM . In other words, if $p(\widehat{x}) = x$, then $\widehat{T_x M}$ is isometric to $T_x M$ through a map I . By Proposition 2.2, $\mathbb{R} \cdot \widehat{X}_1$ is a well defined extension of $\mathbb{R} \cdot X$. Thus, the restriction of $\widehat{\mathcal{N}M}$ to $i(\Lambda \setminus \text{Sing}(X))$ is isomorphic to the normal bundle $\mathcal{N}M|_{\Lambda \setminus \text{Sing}(\Lambda)}$ through the map I .

Finally Theorem 2.7 shows that the fibered flow \widehat{P}^* is conjugated by I near the zero-section to the rescaled sectional Poincaré flow P^* . \square

2.6 Linear Poincaré flow: robustness of the dominated splitting

As we mentioned at the end of Section 2.4, the linear Poincaré flow has been compactified in [LGW] as the normal flow acting on the bundle $\mathcal{N}T^1M$ over T^1M . This allows in some cases to prove the robustness of the dominated splitting of the linear Poincaré flow.

Proposition 2.8. *Let us consider $X \in \mathcal{X}^1(M)$, where $\dim M = 3$, and an invariant compact set Λ such that any singularity $\sigma \in \Lambda$ has real eigenvalues $\lambda_1 < \lambda_2 < 0 < \lambda_3$ and $W^{ss}(\sigma) \cap \Lambda = \{\sigma\}$.*

If the linear Poincaré flow on $\Lambda \setminus \text{Sing}(X)$ has a dominated splitting, then there exist neighborhoods \mathcal{U} of X and U of Λ such that for any $Y \in \mathcal{U}$ and any invariant compact set $\Lambda' \subset U$ for Y , the linear Poincaré flow of Y on $\Lambda' \setminus \text{Sing}(Y)$ has a dominated splitting.

Proof. Let us consider the (continuous) unit tangent flow $U\varphi^X$ associated to X and acting on T^1M . The set $M \setminus \text{Sing}(X)$ embeds by the map $i_X : x \mapsto (x, X(x)/\|X(x)\|)$. We denote by $S(E)$ the set of unit vectors of a vector space E . Thus $S(T_x M) = T_x^1 M$. We introduce the set

$$\Delta_X := i_X(\Lambda \setminus \text{Sing}(X)) \cup \bigcup_{\sigma \in \text{Sing}(X) \cap \Lambda} S(E^{cu}(\sigma)).$$

It is compact (by our assumptions on X at the singularities in Λ) and invariant by $U\varphi$.

Lemma 2.9. *Under the assumptions of the proposition, the closure of $i_Y(\Lambda' \setminus \text{Sing}(Y))$ in T^1M is contained in a small neighborhood of Δ_X .*

Proof. Let us define $B(\Lambda)$ as the set of points $(x, u) \in T^1M$ with $x \in \Lambda$ such that there exists sequences $Y_n \rightarrow X$ in $\mathcal{X}^1(M)$ and $x_n \in M \setminus \text{Sing}(X_n)$ such that

$$-(x_n, Y_n(x_n)/\|Y_n(x_n)\|) \rightarrow (x, u),$$

- the orbit of x_n for the flow of Y_n is contained in the $1/n$ -neighborhood of Λ .

For each $\sigma \in \text{Sing}(X) \cap \Lambda$, we have to show that

$$B(\Lambda) \cap T_\sigma^1 M \subset S(E^{cu}(\sigma)).$$

Now the property we want is exactly the same as [LGW, Lemma 4.4]. The definition of $B(\Lambda)$ and the setting differ but we can apply the same argument: the assumptions of [LGW, Lemma 4.4] can be replaced by that any $\sigma \in \Lambda$ has real eigenvalues $\lambda_1 < \lambda_2 < 0 < \lambda_3$ and $W^{ss}(\sigma) \cap \Lambda = \{\sigma\}$.

At each $\sigma \in \Lambda \cap \text{Sing}(X)$, one considers a chart and one fixes a point z in $W^{ss}(\sigma) \setminus \{\sigma\}$. For each Y close to X , each point y close to the continuation σ_Y and whose orbit $(\varphi_t^Y(y))$ lies in a neighborhood of Λ , let us assume by contradiction that $(y - \sigma_Y)/\|y - \sigma_Y\|$ is not close to the center-unstable plane of the singularity σ_Y of Y . After some backward iteration $\varphi_t(y)$ is still close to σ_Y and $(\varphi_t(y) - \sigma_Y)/\|\varphi_t(y) - \sigma_Y\|$ gets close to the strong stable direction of σ_Y . Iterating further in the past, one gets $\varphi_s(y)$ close to z : the distance $d(\varphi_s(y), z)$ can be chosen arbitrarily small if Y is close enough to X and if y is close enough to σ_Y . Taking the limit, one concludes that z belongs to Λ : this is in contradiction with $W^{ss}(\sigma) \cap \Lambda = \{\sigma\}$. \square

By definition (see Section 2.2), if the linear Poincaré flow for X is dominated on $\Lambda \setminus \text{Sing}(X)$, then the normal flow $\mathcal{N}\varphi^X$ for X is dominated on $i_X(\Lambda \setminus \text{Sing}(X))$. Note that $\mathcal{N}\varphi^X$ is also dominated on $S(E^{cu}(\sigma))$ (the orthogonal projection of the splitting $E^{ss} \oplus E^{cu} \subset T_\sigma M$ on the fibers $\mathcal{N}T_z^1 M$ for $z \in S(E^{cu}(\sigma))$ is invariant). Consequently the dynamics of the linear cocycle $\mathcal{N}\varphi^X$ above the compact set $\Delta_X \subset T^1 M$ is also dominated.

By Lemma 2.9 for Y C^1 -close to X and Λ' in a neighborhood of U , the set $i_Y(\Lambda' \setminus \text{Sing}(Y))$ is contained in a neighborhood of Δ_X ; moreover the linear flow $\mathcal{N}\varphi^Y$ associated to Y is close to $\mathcal{N}\varphi^X$. This shows that the dynamics of $\mathcal{N}\varphi^Y$ above $i_Y(\Lambda' \setminus \text{Sing}(Y))$ is dominated. By definition, this implies that the linear Poincaré flow associated to Y above $\Lambda' \setminus \text{Sing}(Y)$ is dominated. \square

3 Fibered dynamics with a dominated splitting

In this section we introduce an identification structure (for local fibered dynamics) which formalizes the properties satisfied by the rescaled sectional Poincaré flow. We then discuss some consequences of the existence of a dominated splitting inside the fibers.

3.1 Dominated splitting for a local fibered flow

We consider local fibered flows as in Definition 1.2. The following notations will be used.

Notations. – One sometimes denotes a point $u \in \mathcal{N}_x$ as u_x to emphasize the base point x .

– The length of a C^1 curve $\gamma \subset \mathcal{N}_x$ (with respect to the metric of \mathcal{N}_x) is denoted by $|\gamma|$.

– A ball centered at u and with radius r inside a fiber \mathcal{N}_x is denoted by $B(u, r)$.

– For $x \in K$, $t \in \mathbb{R}$ and $u \in \mathcal{N}_x$, one denotes by $DP_t(u)$ the derivative of P_t at u along \mathcal{N}_x . In particular $(DP_t(0_x))_{t \in \mathbb{R}, x \in K}$ defines a linear flow over the 0-section of \mathcal{N} .

Definition 3.1. *The local flow (P_t) admits a dominated splitting $\mathcal{N} = \mathcal{E} \oplus \mathcal{F}$ if \mathcal{E}, \mathcal{F} are subbundles of \mathcal{N} that are invariant by the linear flow $(DP_t(0))$ and if there exists $\tau_0 > 0$ such that for any $x \in K$, for any unit $u \in \mathcal{E}(x)$ and $v \in \mathcal{F}(x)$ and for any $t \geq \tau_0$ we have:*

$$\|DP_t(0_x).u\| \leq \frac{1}{2} \|DP_t(0_x).v\|.$$

Moreover we say that \mathcal{E} is 2-dominated if there exists $\tau_0 > 0$ such that for any $x \in K$, any unit vectors $u \in \mathcal{E}(x)$ and $v \in \mathcal{F}(x)$, and for any $t \geq \tau_0$ we have:

$$\max(\|DP_t(0_x).u\|, \|DP_t(0_x).u\|^2) \leq \frac{1}{2} \|DP_t(0_x).v\|.$$

When there exists a dominated splitting $\mathcal{N} = \mathcal{E} \oplus \mathcal{F}$ and $V \subset K$ is an open subset, one can prove that \mathcal{E} is uniformly contracted by considering the induced dynamics on $K \setminus V$ and checking that the following property is satisfied.

Definition 3.2. *The bundle \mathcal{E} is uniformly contracted on the set V if there exists $t_0 > 0$ such that for any $x \in K$ satisfying $\varphi_t(x) \in V$ for any $0 \leq t \leq t_0$ we have:*

$$\|DP_{t_0}|_{\mathcal{E}(x)}\| \leq \frac{1}{2}.$$

We say that \mathcal{E} is uniformly contracted if it is uniformly contracted on K .

3.2 Identifications

3.2.1 Definition of identifications

We introduce more structures on the fibered dynamics.

Definition 3.3. *A C^k -identification π on an open set $U \subset K$ is defined by two constants $\beta_0, r_0 > 0$ and a continuous family of C^k -diffeomorphisms $\pi_{y,x}: \mathcal{N}_y \rightarrow \mathcal{N}_x$ associated to pairs of points $x, y \in K$ with $x \in U$ and $d(x, y) < r_0$, such that:*

For any $\{x, y, z\}$ of diameter smaller than r_0 with $x, z \in U$ and any $u \in B(0, \beta_0) \subset \mathcal{N}_y$,

$$\pi_{z,x} \circ \pi_{y,z}(u) = \pi_{y,x}(u).$$

In particular $\pi_{x,x}$ coincides with the identity on $B(0, \beta_0)$.

Notations. – We will sometimes denote $\pi_{y,x}$ by π_x . Also the projection $\pi_{y,x}(0) = \pi_x(0_y)$ of $0 \in \mathcal{N}_y$ on \mathcal{N}_x will be denoted by $\pi_x(y)$.

– We will denote by Lip be the set of orientation-preserving bi-Lipschitz homeomorphisms θ of \mathbb{R} (and by $\text{Lip}_{1+\rho}$ the set of those whose Lipschitz constant is smaller than $1 + \rho$).

Definition 3.4. *The identification π on U is compatible with the flow (P_t) if:*

1. Transverse boundary. *For any segment of orbit $\{\varphi_s(x), s \in [-t, t]\}$, $t > 0$, contained in $K \setminus U$ we have $x \in K \setminus \overline{U}$.*
2. No small period. *For any $\varkappa > 0$, there is $r > 0$ such that for any $x \in \overline{U}$ and $t \in [-2, 2]$ with $d(x, \varphi_t(x)) < r$, then we have $|t| < \varkappa$.*
3. Local injectivity. *For any $\delta > 0$, there exists $\beta > 0$ such that for any $x, y \in U$: if $d(x, y) < r_0$ and $\|\pi_x(y)\| \leq \beta$, then $d(\varphi_t(y), x) \leq \delta$ for some $t \in [-1/4, 1/4]$.*
4. Local invariance. *For any $x, y \in U$ and $t \in [-2, 2]$ such that y and $\varphi_t(y)$ are r_0 -close to x , and for any $u \in B(0, \beta_0) \subset \mathcal{N}_y$, we have*

$$\pi_x \circ P_t(u) = \pi_x(u).$$

5. Global invariance. *For any $\delta, \rho > 0$, there exists $r, \beta > 0$ such that:*

For any $y, y' \in K$ with $y \in U$ and $d(y, y') < r$, for any $u \in \mathcal{N}_y$, $u' \in \mathcal{N}_{y'}$ with $\pi_y(u') = u$, and any intervals $I, I' \subset \mathbb{R}$ containing 0 and satisfying

$$\|P_s(u)\| < \beta \text{ and } \|P_{s'}(u')\| < \beta \text{ for any } s \in I \text{ and any } s' \in I',$$

there is $\theta \in \text{Lip}_{1+\rho}$ such that $\theta(0) = 0$ and $d(\varphi_s(y), \varphi_{\theta(s)}(y')) < \delta$ for any $s \in I \cap \theta^{-1}(I')$. Moreover, for any $v \in \mathcal{N}_y$ such that $\|P_s(v)\| < \beta$ for each $s \in I \cap \theta^{-1}(I')$ then

- $v' = \pi_{y'}(v)$ satisfies $\|P_{\theta(s)}(v')\| < \delta$ for each such s ,
- if $\varphi_s(y) \in U$ for some s , then $\pi_{\varphi_s(y)} \circ P_{\theta(s)}(v') = P_s(v)$.

Remarks 3.5. a) These definitions are still satisfied if one reduces r_0 or β_0 . Their value will be reduced in the following sections in order to satisfy additional properties.

One can also rescale the time and keep a compatible identification: the flow $t \mapsto \varphi_{t/C}$ for $C > 1$ still satisfies the definitions above, maybe after reducing the constant r_0 .

The main point to check is that the time t in the Local injectivity can still be chosen in $[-1/4, 1/4]$. Indeed, this is ensured by the “No small period” assumption applied with $\varkappa = 1/4C$: if r_0 is chosen smaller and if both $d(\varphi_t(y), x), d(y, x)$ are less than r_0 for $t \in [-1/4, 1/4]$, then $|t|$ is smaller than \varkappa . Now the time t in the Local injectivity property belongs to $[-\varkappa, \varkappa]$ for the initial flow, hence to $[-1/4, 1/4]$ for the time-rescaled flow.

b) The “No small period” assumption (which does not involve the projections π_x) is equivalent to the non-existence of periodic orbits of period ≤ 2 which intersect \overline{U} . In particular, by reducing r_0 , one can assume the following property:

For any $x \in U$ and any $t \in [1, 2]$, we have $d(x, \varphi_t(x)) \geq r_0$.

c) For $x \in U$, the Local injectivity prevents the existence of $y \in U$ that is r_0 -close to x , is different from $\varphi_t(x)$ for any $t \in [-1/4, 1/4]$, and such that $\pi_x(y) = 0_x$. In particular:

If $x, \varphi_t(x) \in U$ and $t \notin (-1/2, 1/2)$ satisfy $\pi_x(\varphi_t(x)) = 0_x$, then x is periodic.

d) The Global invariance says that when two orbits $(P_s(u))$ and $(P_s(u'))$ of the local fibered flow are close to the 0 section of \mathcal{N} and have two points which are identified by π , then they are associated to orbits of the flow φ that are close (up to a reparametrization θ). In this case, any orbit of (P_t) close to the zero-section above the first φ -orbit can be projected to an orbit of (P_t) above the second φ -orbit.

e) The Global invariance can be applied to pairs of points y, y' where the condition $d(y, y') < r$ has been replaced by a weaker one $d(y, y') < r_0$. In particular, this gives:

For any $\delta, \rho > 0$, there exist $\beta > 0$ such that: if $y, y' \in K$, $u \in \mathcal{N}_y$, $u' \in \mathcal{N}_{y'}$ and the intervals $I, I' \subset \mathbb{R}$ containing 0 satisfy

- $d(y, y') < r_0$ and $y \in U$,
- $\pi_y(u') = u$, $\|P_s(u)\| < \beta$ and $\|P_{s'}(u')\| < \beta$ for any $s \in I$ and any $s' \in I'$,

there is $\theta \in \text{Lip}_{1+\rho}$ such that $|\theta(0)| \leq 1/4$ and $d(\varphi_s(y), \varphi_{\theta(s)}(y')) < \delta$ for any $s \in I \cap \theta^{-1}(I')$.

Indeed provided that $\beta > 0$ has been chosen small enough, one can apply the Local injectivity and the Local invariance in order to replace y' and u' by $y'' = \varphi_t(y')$ and $u'' = P_s(u')$ for some $t \in [-1/4, 1/4]$ such that $d(y, y'') < r$. The assumptions for the Global invariance then are satisfied by y, y'' and u, u'' . It gives a $\theta \in \text{Lip}_{1+\rho}$ satisfying $d(\varphi_s(y), \varphi_{\theta(s)}(y')) < \delta$ for $s \in I \cap \theta^{-1}(I')$ but the condition $\theta(0) = 0$ has been replaced by $\theta(0) = t$; in particular $|\theta(0)| < 1/4$.

Fundamental example. One may think that (φ_t) is the compactified flow $\widehat{\varphi}$ on an invariant compact set $K \subset \widehat{M}$ as in Section 2, that \mathcal{N} is the compactified normal bundle over K , and that (P_t) is the extended rescaled sectional Poincaré flow.

3.2.2 No shear inside orbits

The next property states that one cannot find two reparametrizations of a same orbit, that shadow each other, coincide for some parameter and differ by at least 2 for another parameter.

Proposition 3.6. *If r_0 is small enough, for any $x \in U$, any increasing homeomorphism θ of \mathbb{R} , any interval I containing 0 satisfying $\varphi_{\theta(0)}(x) \in U$ and $d(\varphi_t(x), \varphi_{\theta(t)}(x)) \leq r_0$, $\forall t \in I$, then:*

- $\theta(0) > 1/2$ implies that $\theta(t) > t + 2$, $\forall t \in I$ such that $\varphi_t(x), \varphi_{\theta(t)}(x) \in U$;
- $\theta(0) \in [-2, 2]$ implies that $\theta(t) \in [t - 1/2, t + 1/2]$, $\forall t \in I$ such that $\varphi_t(x), \varphi_{\theta(t)}(x) \in U$;
- $\theta(0) < -1/2$ implies that $\theta(t) < t - 2$, $\forall t \in I$ such that $\varphi_t(x), \varphi_{\theta(t)}(x) \in U$.

Proof. Let Δ be the set of points x such that $\varphi_t(x) \notin U$ for every $|t| \leq 1/2$. By the “Transverse boundary” assumption (and up to reduce r_0), it is compact and its distance to \overline{U} is larger than $2r_0$. Let us choose $\varepsilon \in (0, 1/2)$ small enough so that $\varphi_s(\overline{U})$ is disjoint from the r_0 -neighborhood of Δ when $|s| \leq \varepsilon$. Still reducing r_0 , one can assume that:

- (a) any piece of orbit $\{\varphi_s(y), s \in [0, b]\} \subset K \setminus U$, with $y, \varphi_b(y)$ in the r_0 -neighborhood of Δ and $b \leq 1/2$, is disjoint from the r_0 -neighborhood of \overline{U} ,
- (b) if $\varphi_s(y)$ is r_0 -close to $y \in \overline{U}$ for $|s| \leq 2$, then $|s| \leq \varepsilon$.

The first condition is satisfied by small $r_0 > 0$ since otherwise letting $r_0 \rightarrow 0$ one would construct $y, \varphi_b(y) \in \Delta$ and $\varphi_s(y) \in \overline{U}$ where $0 \leq s \leq b \leq 1/2$, contradicting the definition of Δ . The second condition is a consequence of the “No small period” assumption.

Claim. *If $\theta(0) \geq -2$ then $\theta(t) \geq t - 1/2$ for any $t \in I$ satisfying $\varphi_t(x) \in U$.*

Proof. The case $t = 0$ is a consequence of the “No small period” assumption. We deal with the case that t is positive. The case that t is negative can be deduced by applying the positive case to θ^{-1} and to the point $\varphi_{\theta(0)}(x)$.

Let J be the interval of $t \in I$ satisfying for all $s \in J$ one has either $\varphi_s(x) \notin \overline{U}$, or $\theta(s) \geq s - 1/2$. Let $t_1 \in [0, t_0] \cap J$ be the largest time satisfying $\varphi_{t_1}(x) \in \overline{U}$. It exists since if (t_k) is an increasing sequence in $[0, t_0] \cap J$ satisfying $\varphi_{t_k}(x) \in \overline{U}$, then we have $\theta(t_k) \geq t_k - 1/2$. So the limit \bar{t} satisfies $\theta(\bar{t}) \geq \bar{t} - 1/2$ (and belongs to J) and $\varphi_{\bar{t}}(x) \in \overline{U}$.

By property (b), $\theta(t_1) \geq t_1 - \varepsilon$. For $s > t_1$ close to t_1 , we thus have $s \in J$. Since t_1 is maximal we also get $\varphi_s(x) \notin \overline{U}$. Since $\varphi_{t_0}(x) \in U$, there exists a minimal $t_2 \in [t_1, t_0]$ such that $\varphi_{t_2}(x)$ belongs to the boundary of U . In particular $\varphi_{\theta(t_2)}(x)$ is r_0 -close to the boundary of U . Note that $[t_1, t_2] \subset J$. By maximality of t_1 one has $\theta(t_2) < t_2 - 1/2$.

The “No small period” assumption implies $\theta(t_2) < t_2 - 2$. In particular,

$$t_2 > \theta(t_2) + 2 > \theta(t_1) + 2 \geq t_1 + 2 - \varepsilon.$$

This shows that $\varphi_s(x) \in \Delta$, $\forall s \in [t_1 + 1/2, t_2 - 1/2]$. Since $\varphi_{\theta(t_1+1/2)}(x)$ is r_0 -close to $\varphi_{t_1+1/2}(x) \in \Delta$, and $\varphi_{t_1}(x) \in \overline{U}$, one has $|\theta(t_1 + 1/2) - t_1| > \varepsilon$ (by our choice of ε). Since $\theta(t_1) \geq t_1 - \varepsilon$, this gives $\theta(t_1 + 1/2) > t_1 + \varepsilon$. Hence $(\theta(t_1 + 1/2), t_1 + 1/2]$ has length smaller than $1/2$.

If $\theta(t_2) \in (\theta(t_1 + 1/2), t_1 + 1/2]$, since $\varphi_{\theta(t_1+1/2)}(x)$ belongs to the r_0 -neighborhood of Δ and since $\varphi_{t_1+1/2}(x) \in \Delta$, the property (a) implies that $\varphi_{\theta(t_2)}(x)$ is disjoint from the r_0 -neighborhood of \overline{U} . Otherwise $\theta(t_2) \in [t_1 + 1/2, t_2 - 1/2]$ and then $\varphi_{\theta(t_2)}(x) \in \Delta$ is r_0 far from \overline{U} . This is a contradiction since we have proved before that $\varphi_{\theta(t_2)}(x)$ is r_0 -close to the boundary of U . \square

Arguing in a same way we deduce that $\theta(0) \leq 2$ implies $\theta(t) \leq t + 1/2$ for any $t \in I$ such that $\varphi_t(x) \in U$. One deduces the second item of the proposition. Note that if $\theta(t_0) \leq t_0 + 2$ for some t_0 with $\varphi_{t_0}(x) \in U$, then one deduces that $\theta(t) \leq t + 1/2$ for all other t and in particular $\theta(0) \leq 2$; this gives the first item. The third one is similar. \square

3.2.3 Closing lemmas

The following closing lemma is an example of properties given by identifications.

Lemma 3.7. *Let us assume that β_0, r_0 are small enough. Let us consider:*

- $x \in U$ having an iterate $y = \varphi_T(x)$ in $U \cap B(x, r_0)$ with $T \geq 4$,
- a fixed point $p \in \mathcal{N}_x$ for $\tilde{P}_T := \pi_x \circ P_T$ such that $\|P_t(p)\| < \beta_0$ for each $t \in [0, T]$,
- a sequence (y_k) in a compact set of $U \cap B(x, r_0/2)$ such that $\pi_x(y_k)$ converges to p .

Then there exists a sequence (s_k) in $[-1, 1]$ such that $\varphi_{s_k}(y_k)$ converges to a periodic point y of K having some period T' such that $\pi_x(y) = p$ and

$$DP_{T'}(0_y) = D\pi_x(0_y)^{-1} \circ D\tilde{P}_T(p) \circ D\pi_x(0_y).$$

Proof. Up to extract a subsequence, (y_k) converges to a point $y \in U \cap B(x, r_0/2)$ such that $\pi_x(y) = p$. By the Local injectivity, since the sequence $(\pi_y(y_k))$ converges to 0_y , there exists (s_k) in $[-1, 1]$ such that $\varphi_{s_k}(y_k)$ converges to y .

By the Global invariance, there exists (T_k) satisfying $\frac{1}{2}T \leq T_k \leq 2T$ such that $\varphi_{T_k}(y_k)$ is in $B(x, r_0/2)$ and projects by π_x on $\tilde{P}_T(\pi_x(y_k))$.

In particular $(\pi_x \circ \varphi_{T_k}(y_k))$ converges to p and $(\pi_y \circ \varphi_{T_k}(y_k))$ converges to 0_y . One deduces (up to modify T_k by adding a real number in $[-1, 1]$) that $\varphi_{T_k}(y_k)$ converges to y . Since $T \geq 4$, the limit value T' of T_k is larger than 1 and one deduces that y is T' -periodic. Moreover, $\pi_x(y) = p$ so that by the Global invariance $D\tilde{P}_T(p)$ and $DP_{T'}(0_y)$ are conjugated by $D\pi_x(0_y)$. \square

For the next statement, we consider an open set V containing $K \setminus U$ so that points in $K \setminus V$ are separated from the boundary of U by a distance much larger than r_0 .

Corollary 3.8. *Assume that β_0, r_0 are small enough. If $x \in K \setminus V$ has an iterate $y = \varphi_T(x)$ in $B(x, r_0)$ with $T \geq 4$ and if there exists a subset $B \subset \mathcal{N}_x$ containing 0 such that*

- $P_t(B) \subset B(0_{\varphi_t(x)}, \beta_0)$ for any $0 < t < T$,
- $\tilde{P}_T := \pi \circ P_T$ sends B into itself,
- the sequence $\tilde{P}_T^k(0)$ converges to a fixed point $p \in B$,

then the positive orbit of x by φ also converges to a periodic orbit.

Proof. From the Global invariance, there exists a sequence $T_k \rightarrow +\infty$ such that $y_k := \varphi_{T_k}(x)$ projects by π_x on $\tilde{P}_T^k(0)$ and $|T_{k+1} - T_k|$ is uniformly bounded in k .

Since $(\tilde{P}_T^k(0) = \pi_x(y_k))$ converges to p , we can apply the previous lemma so that $(\varphi_{s_k}(y_k))$ converges to a T' -periodic point $y \in K$ for some $s_k \in [-1, 1]$. Since $|T_{k+1} - T_k|$ is uniformly bounded in k , this proves that the ω -limit set of x is the orbit of y . \square

3.2.4 Generalized orbits

The identifications π allow us to introduce generalized orbits. In the case where K is a non-singular invariant set and (P_t) is the sectional Poincaré flow on \mathcal{N} , these orbits correspond to the orbits of the flow contained in the maximal invariant set in a neighborhood of K .

Definition 3.9 (Generalized orbit). *A (piecewise continuous) path $\bar{u} = (u(t))_{t \in \mathbb{R}}$ in \mathcal{N} is a generalized orbit if there is a sequence $(t_n)_{n \in \mathbb{Z}}$ in \mathbb{R} such that if $y(t)$ denotes the projection of $u(t)$ to K by the bundle map $\mathcal{N} \rightarrow K$, then for each $n \in \mathbb{Z}$:*

- $t_{n+1} - t_n \geq 1$,
- $\|u(t)\| < \beta_0$ and $u(t) = P_{t-t_n}(u_n)$ for $t \in [t_n, t_{n+1})$,
- $\varphi_{t_{n+1}-t_n}(y_n), y_{n+1}$ belong to U , are r_0 -close and $\pi_{y_{n+1}}(P_{t_{n+1}-t_n}(u_n)) = u_{n+1}$.

The projection of $u(t)$ from \mathcal{N} to K defines a pseudo-orbit $(y(t))$ of φ in K .

Remarks 3.10. a) If $(u(t))$ is a generalized orbit, then $(u(t+s))_{t \in \mathbb{R}}$ is also for any $s \in \mathbb{R}$.

b) The generalized orbits satisfying $u(t) = 0_{y(t)}$ for any t can be identified to the orbits of φ on K which meet U for arbitrarily large positive and negative times t_n .

Definition 3.11 (Topology on generalized orbits). Let us fix \bar{u} . For $T > 0$ large and $\eta > 0$ small, we say that a generalized orbit \bar{u}' is (T, η) -close to \bar{u} if $u(t)$ and $u'(t)$ are η -close for each $t \in [-T, T]$.

For the next notion, we fix an open set V containing $K \setminus U$.

Definition 3.12 (Neighborhood of K). A generalized orbit belongs to the η -neighborhood of K (or of the 0-section of \mathcal{N}) if the additional conditions hold:

- $d(y(t_{n+1}), \varphi_{t_{n+1}-t_n}(y(t_n))) \leq \eta$, for any $n \in \mathbb{Z}$,
- $d(y(t_n), K \setminus V) < \eta$, for each $n \in \mathbb{Z}$ such that $y(t_n) \neq \varphi_{t_n-t_{n-1}}(y(t_{n-1}))$,
- $\|u(s)\| < \eta$ for any $s \in \mathbb{R}$.

Definition 3.13 (Generalized flow). We associate, to any generalized orbit $\bar{u} = (u(t))$ and any $s, t \in \mathbb{R}$, a diffeomorphism \bar{P}_t from a neighborhood of $u(s)$ in $\mathcal{N}_{y(s)}$ to a neighborhood of $u(s+t)$ in $\mathcal{N}_{y(s+t)}$ which for any t, t' satisfies $\bar{P}_{t'} \circ \bar{P}_t = \bar{P}_{t+t'}$. It is defined by:

- by $\bar{P}_t = P_t$ when $t_n \leq s \leq t + s < t_{n+1}$,
- by $\bar{P}_t = P_{t+s-t_{n+1}} \circ \pi_{y(t_{n+1})} \circ P_{t_{n+1}-s}$ when $t_n \leq s < t_{n+1} \leq t + s < t_{n+2}$,
- and by applying inductively the flow relation in the other cases.

The generalized flow acts on generalized orbits: $\bar{P}_t(\bar{u})$ coincides at time s with $\bar{u}(s+t)$. When \bar{u} can be identified to an orbit of φ (as in Remark 3.10), \bar{P}_t coincides with the flow P_t .

Half generalized orbits. The previous definitions may be extended to any (piecewise continuous) path $(u(t))_{t \in I}$ parametrized by an interval I of \mathbb{R} . When $I = [0, +\infty)$ or $(-\infty, 0]$ one gets the notion of half generalized orbits. The generalized semi-flow $(\bar{P}_t)_{t \geq 0}$ (resp. $(\bar{P}_t)_{t \leq 0}$) acts on half generalized orbits parametrized by $[0, +\infty)$ (resp. $(-\infty, 0]$).

3.2.5 Normally expanded irrational tori

We give a setting of a dominated splitting $\mathcal{E} \oplus \mathcal{F}$ such that \mathcal{E} is not uniformly contracted.

Definition 3.14. A normally expanded irrational torus is an invariant compact subset $\mathcal{T} \subset K$ such that

- the dynamics of $\varphi|_{\mathcal{T}}$ is topologically equivalent to an irrational flow on \mathbb{T}^2 ,
- there exists a dominated splitting $\mathcal{N}|_{\mathcal{T}} = \mathcal{E} \oplus \mathcal{F}$ and \mathcal{E} has one-dimensional fibers,
- for some $x \in U \cap \mathcal{T}$ and $r > 0$, $\pi_x(\{z \in K, d(x, z) < r\})$ is a C^1 -curve tangent to $\mathcal{E}(x)$.

The name is justified as follows.

Lemma 3.15. *For any normally expanded irrational torus \mathcal{T} , the Lyapunov exponent along \mathcal{E} of the (unique) invariant measure of φ on \mathcal{T} is equal to zero; in particular \mathcal{F} is uniformly expanded (i.e uniformly contracted by backward iterations).*

Remark 3.16. *With the technics of Section 4, one can also prove that the α -limit set of any point z in a neighborhood coincides with \mathcal{T} .*

Proof. Let us choose a global transversal $\Sigma \simeq \mathbb{T}^1$ containing x for the restriction of φ to \mathcal{T} . The dynamics is conjugated to a suspension of an irrational rotation of Σ . We consider the sequence (t_k) of positive returns of the orbit of x inside a neighborhood of x in Σ . Note that $|t_{k+1} - t_k|$ is uniformly bounded. For every $y \in \mathcal{T}$ close to x there exists a sequence (t'_k) such that $|t_{k+1} - t_k|$ and $|t'_{k+1} - t'_k|$ are close and $\varphi_{t'_k}(y)$ is close to $\varphi_{t_k}(x)$ and belongs to Σ . In particular by choosing y close enough to x , there exist $\varepsilon_1, \varepsilon_2 > 0$ arbitrarily small such that

$$\varepsilon_1 \leq d(\varphi_{t'_k}(y), \varphi_{t_k}(x)) \leq \varepsilon_2.$$

Let $I \subset \Sigma$ be the interval bounded by x, y . The transversal Σ is mapped homeomorphically inside an interval of the C^1 -curve $\gamma = \pi_x(\{z \in K, d(x, z) < r\})$ and by the Global invariance $\pi_x \circ P_{t_k}$ sends $\pi_x(y)$ to $\pi_x(\varphi_{t'_k}(y))$ and similarly $\pi_x(I) \subset \gamma$ inside γ . Moreover, there exists $\varepsilon'_1, \varepsilon'_2 > 0$ arbitrarily small such that for each k ,

$$\varepsilon'_1 \leq |\pi_x \circ P_{t_k} \circ \pi_x(I)| \leq \varepsilon'_2.$$

Since $t_{k+1} - t_k$ is bounded, it implies that the Lyapunov exponent along \mathcal{E} vanishes. \square

3.2.6 Contraction on periodic orbits and criterion for 2-domination

When there exists a dominated splitting $\mathcal{E} \oplus \mathcal{F}$ where \mathcal{E} is one-dimensional, the uniform contraction of the bundle \mathcal{E} above each periodic orbit of K , implies that it is 2-dominated.

Proposition 3.17. *Let us assume that*

- *there exists a dominated splitting $\mathcal{N} = \mathcal{E} \oplus \mathcal{F}$ and the fibers of \mathcal{E} are one-dimensional,*
- *\mathcal{E} is uniformly contracted on an open set V containing $K \setminus U$.*

Then either the bundle \mathcal{E} is 2-dominated, or there exists a periodic orbit \mathcal{O} in K whose Lyapunov exponents are all positive.

Proof. If there is no 2-domination, there exists a sequence (x_n) in K , such that

$$\|DP_n|_{\mathcal{E}(x_n)}\|^2 \geq \frac{1}{2} \|DP_n|_{\mathcal{F}(x_n)}\|.$$

One can extract a φ -invariant measure from the sequence

$$\mu_n := \frac{1}{n} \int_{t=0}^n \delta_{\varphi_t(x_n)} dt$$

and the maximal Lyapunov exponents $\lambda^{\mathcal{E}}, \lambda^{\mathcal{F}}$ along \mathcal{E}, \mathcal{F} satisfy $2\lambda^{\mathcal{E}} \geq \lambda^{\mathcal{F}}$. In particular, $\lambda^{\mathcal{F}} > \lambda^{\mathcal{E}} \geq \lambda^{\mathcal{F}} - \lambda^{\mathcal{E}} > 0$. Since \mathcal{E} is one-dimensional, one deduces that φ admits an ergodic measure μ whose Lyapunov exponents are both positive. Since \mathcal{E} is uniformly contracted on V , the support of μ has to intersect U . For μ -almost every point $x \in K$, there exists a neighborhood V_x of 0 in \mathcal{N}_x such that $\|(DP_{-t})|_{V_x}\|$ decreases exponentially as $t \rightarrow +\infty$. In particular, one can take $x \in U$ recurrent and find a large time $T > 0$ such that $\tilde{P}_{-T} := \pi_x \circ P_{-T}$ sends V_x into itself as a contraction. By Corollary 3.8, there is a periodic point y in K with some period $T' > 0$ and a fixed point $p \in V_x$ for \tilde{P}_{-T} such that the tangent map $DP_{-T'}(0_y)$ is conjugate to the tangent map $D\tilde{P}_{-T}(p)$ (by Lemma 3.7). Hence the Lyapunov exponents of y are all positive. \square

3.3 Plaque families

We now introduce center-stable plaques $\mathcal{W}^{cs}(x)$ that are candidates to be the local stable manifolds of the dynamics tangent to \mathcal{E} . A symmetric discussion gives center-unstable plaques \mathcal{W}^{cu} tangent to \mathcal{F} .

3.3.1 Standing assumptions

In this Section 3.3, we consider:

- a bundle \mathcal{N} with d -dimensional fibers, a local fibered flow (\mathcal{N}, P) over a topological flow (φ, K) and an identification π on an open set U , compatible with (P_t) ,
- a dominated splitting $\mathcal{N} = \mathcal{E} \oplus \mathcal{F}$ of the bundle \mathcal{N} ,
- an open set V containing $K \setminus U$.

Reducing r_0 we assume that the distance between $K \setminus V$ and $K \setminus U$ is much larger than r_0 .

We also fix an integer $\tau_0 \geq 1$ satisfying Definition 3.1 of the domination. We choose $\lambda > 1$ such that $\lambda^{4\tau_0} < 2$. In particular for any $x \in K$, for any unit $u \in \mathcal{E}(x)$ and $v \in \mathcal{F}(x)$, we have:

$$\forall t \geq \tau_0, \quad \|DP_t(0).u\| \leq \lambda^{-2t} \|DP_t(0).v\|. \quad (7)$$

In each space $\mathcal{N}_x = \mathcal{E}(x) \oplus \mathcal{F}(x)$, $x \in K$, we introduce the constant cone

$$\mathcal{C}^{\mathcal{E}}(x) := \{u = u^{\mathcal{E}} + u^{\mathcal{F}} \in \mathcal{N}_x = \mathcal{E}(x) \oplus \mathcal{F}(x), \|u^{\mathcal{E}}\| > \|u^{\mathcal{F}}\|\},$$

and $\mathcal{C}^{\mathcal{F}}(x)$ in a symmetric way. They vary continuously with x . Moreover the dominated splitting implies that for any $t \geq \tau_0$ the cone fields are contracted:

$$DP_t(0_x).\overline{\mathcal{C}^{\mathcal{F}}(x)} \subset \mathcal{C}^{\mathcal{F}}(\varphi_t(x)) \text{ and } DP_{-t}(0_x).\overline{\mathcal{C}^{\mathcal{E}}(x)} \subset \mathcal{C}^{\mathcal{E}}(\varphi_{-t}(x)).$$

3.3.2 Plaque family for fibered flows

Definition 3.18. A C^k -plaque family tangent to \mathcal{E} is a continuous fibered embedding $\psi \in \text{Emb}^k(\mathcal{E}, N)$, that is a family of C^k -diffeomorphisms onto their image $\psi_x: \mathcal{E}(x) \rightarrow \mathcal{N}_x$ such that $\psi_x(0_x) = 0_x$, the image of $D\psi_x(0_x)$ coincides with $\mathcal{E}(x)$ and such that ψ_x depends continuously on $x \in K$ for the C^k -topology.

For $\alpha > 0$, we denote by $\mathcal{W}_{\alpha}^{cs}(x)$ the ball centered at 0_x and of radius α inside $\psi_x(\mathcal{E}(x))$ with respect to the restriction of the metric on \mathcal{N}_x and we denote by $\mathcal{W}^{cs}(x) = \mathcal{W}_1^{cs}(x)$.

The plaque family ψ is locally invariant by the time-one map of the flow (P_t) if there exist $\alpha_{\mathcal{E}} > 0$ such that for any $x \in K$ we have

$$P_1(\mathcal{W}_{\alpha_{\mathcal{E}}}^{cs}(x)) \subset \mathcal{W}^{cs}(\varphi_1(x)).$$

Hirsch-Pugh-Shub's plaque family theorem [HPS] generalizes to local fibered flows.

Theorem 3.19. For any local fibered flow (\mathcal{N}, P) admitting a dominated splitting $\mathcal{N} = \mathcal{E} \oplus \mathcal{F}$ there exists a C^1 -plaque family tangent to \mathcal{E} which is locally invariant by P_1 .

If the flow is C^2 and if \mathcal{E} is 2-dominated, then the plaque family can be chosen C^2 .

3.3.3 Plaque family for generalized orbits

The previous result extends to generalized orbits.

Theorem 3.20. *For any local fibered flow (\mathcal{N}, P) admitting a compatible identification and a domination $\mathcal{N} = \mathcal{E} \oplus \mathcal{F}$, there exists $\eta, \alpha_{\mathcal{E}} > 0$ with the following property.*

For any $x \in K$ and any half generalized orbit $\bar{u} = (u(t))_{t \in [0, +\infty)}$ in the η -neighborhood of the zero-section with $u(0) \in \mathcal{N}_x$, there exists a C^1 -diffeomorphism onto its image $\psi_{\bar{u}}: \mathcal{E}(x) \rightarrow \mathcal{N}_x$ such that $\psi_{\bar{u}}(0_x) = u(0)$ and the image of $D\psi_{\bar{u}}(0_x)$ is contained in the cone $\mathcal{C}^{\mathcal{E}}(x)$. Moreover $\psi_{\bar{u}}$ depends continuously for the C^1 -topology on \bar{u} . Denote by $\mathcal{W}^{cs}(\bar{u}) = \psi_{\bar{u}}(\mathcal{E}(x))$.

The family of plaques is locally invariant by \bar{P}_1 :

$$\bar{P}_1(\mathcal{W}_{\alpha_{\mathcal{E}}}^{cs}(\bar{u})) \subset \mathcal{W}^{cs}(\bar{P}_1(\bar{u})),$$

where $\mathcal{W}_{\alpha_{\mathcal{E}}}^{cs}(\bar{u})$ denotes as before the ball centered at $u(0)$ and of radius $\alpha_{\mathcal{E}}$.

When the identification and the flow are C^2 and when \mathcal{E} is 2-dominated, then the plaques can be chosen C^2 and the family $\psi_{\bar{u}}$ depends continuously on \bar{u} for the C^2 -topology.

The tangent space to $\mathcal{W}^{cs}(\bar{u})$ at \bar{u} in \mathcal{N}_x is denoted by $\mathcal{E}(\bar{u})$. By construction it varies continuously with \bar{u} and coincides with $\mathcal{E}(x)$ when \bar{u} is the half orbit $(0_{\varphi_t(x)})_{t \geq 0}$.

Remark. *We have $D\bar{P}_t(\mathcal{E}(\bar{u})) = \mathcal{E}(\bar{P}_t(\bar{u}))$ for any $t \in \mathbb{R}$. Indeed, this holds for $t \in \mathbb{N}$ by local invariance of the plaque family. The forward invariance of $\mathcal{C}^{\mathcal{F}}$ implies that $\bar{P}_t(\mathcal{E}(\bar{u}))$ is tangent to $\mathcal{C}^{\mathcal{E}}$ for any $t \geq 0$. The dominated splitting implies that any vector v tangent to \mathcal{N}_x at \bar{u} whose forward iterates are all tangent to $\mathcal{C}^{\mathcal{E}}$ belongs to $\mathcal{E}(\bar{u})$ (this can be also seen from the sequence of diffeomorphisms introduced in the next section). This characterization implies the invariance of \mathcal{E} for any time.*

3.3.4 Plaque family for sequences of diffeomorphisms

The proofs of Theorems 3.19 and 3.20 are very similar to [HPS, Theorem 5.5]. It is a consequence of a more general result that we state now. We denote by $d = d^{\mathcal{E}} + d^{\mathcal{F}}$ the dimensions of the fibers of $\mathcal{N}, \mathcal{E}, \mathcal{F}$ and endow \mathbb{R}^d with the standard euclidean metric. For $\chi > 0$ let us define the horizontal cone

$$\mathcal{C}_{\chi} = \{(x, y) \in \mathbb{R}^{d^{\mathcal{E}}} \times \mathbb{R}^{d^{\mathcal{F}}}, \chi \|x\| \geq \|y\|\}.$$

Definition 3.21. *A sequence of C^k -diffeomorphisms of \mathbb{R}^d bounded by constants $\beta, C > 0$ is a sequence \underline{f} of diffeomorphisms $f_n: U_n \rightarrow V_n$, $n \in \mathbb{N}$, where $U_n, V_n \subset \mathbb{R}^d$ contain $B(0, \beta)$, such that $f_n(0) = 0$ and such that the C^k -norms of f_n, f_n^{-1} are bounded by C .*

We denote by $\sigma(\underline{f})$ the shifted sequence $(f_{n+1})_{n \geq 0}$ associated to $\underline{f} = (f_n)_{n \geq 0}$.

The sequence has a dominated splitting if there exists $\tau_0 \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ and for any $z \in B(0, \beta) \cap f_n^{-1}(B(0, \beta)) \cap \dots \cap f_{n+\tau_0-1}^{-\tau_0}(B(0, \beta))$, the cone \mathcal{C}_1 is mapped by $D(f_{n+\tau_0-1} \circ \dots \circ f_n)^{-1}(z)$ inside the smaller cone $\mathcal{C}_{1/2}$.

The center stable direction of the dominated splitting is 2-dominated if there exists $\tau_0 \in \mathbb{N}$ such that for any $z \in B(0, \beta) \cap f_n^{-1}(B(0, \beta)) \cap \dots \cap f_{n+\tau_0-1}^{-\tau_0}(B(0, \beta))$ and for any unit vectors u, v satisfying $D(f_{n+\tau_0-1} \circ \dots \circ f_n)(z).u \in \mathcal{C}_1$ and $v \in \mathbb{R}^d \setminus \mathcal{C}_1$, then

$$\|D(f_{n+\tau_0-1} \circ \dots \circ f_n)(z).u\|^2 \leq \frac{1}{2} \|D(f_{n+\tau_0-1} \circ \dots \circ f_n)(z).v\|.$$

Theorem 3.22. *For any C, β, τ_0 , there exists $\alpha \in (0, \beta)$, and for any sequence of C^1 -diffeomorphisms $\underline{f} = (f_n)$ of \mathbb{R}^d bounded by β, C with a dominated splitting, associated to the constant τ_0 , there exists a C^1 -map $\psi = \psi(\underline{f}): \mathbb{R} \rightarrow \mathbb{R}$ such that:*

- For any z, z' in the graph $\{(x, \psi(\underline{F})(x)), x \in \mathbb{R}^{d^\varepsilon}\}$, the difference $z' - z$ is contained in $\mathcal{C}_{\frac{1}{2}}$.
- (Local invariance.) $f_0(\{(x, \psi(\underline{F})(x)), |x| < \alpha\}) \subset \{(x, \psi(\sigma(\underline{F}))(x)), x \in \mathbb{R}^{d^\varepsilon}\}$.
- The function ψ depends continuously on \underline{F} for the C^1 -topology: for any $R, \varepsilon > 0$, there exists $N \geq 1$ and $\delta > 0$ such that if the two sequences \underline{F} and \underline{F}' satisfy

$$\|(f_n - f'_n)|_{B(0, \beta)}\|_{C^1} \leq \delta \text{ for } 0 \leq n \leq N$$

then $\|(\psi(\underline{F}) - \psi(\underline{F}'))|_{B(0, R)}\|_{C^1}$ is smaller than ε .

- For sequences of C^2 -diffeomorphisms \underline{F} such that the center stable direction of the dominated splitting is 2-dominated (still for the constant τ_0), the function $\psi(\underline{F})$ is C^2 and depends continuously on \underline{F} for the C^2 -topology.

The proof of this theorem is standard. It is obtained by

- introducing a sequence of diffeomorphisms (\widehat{f}_n) defined on the whole plane \mathbb{R}^d which coincide with the diffeomorphisms f_n on a uniform neighborhood of 0 and with the linear diffeomorphism $Df_n(0)$ outside a uniform neighborhood of 0,
- applying a graph transform argument.

Theorem 3.19 is a direct consequence of Theorem 3.22: for each $x \in K$, we consider the sequence of local diffeomorphisms $P_1: \mathcal{N}_{\varphi_n(x)} \rightarrow \mathcal{N}_{\varphi_{n+1}(x)}$. There are bounded linear isomorphisms which identify $\mathcal{N}_{\varphi_n(x)}$ with \mathbb{R}^d and send the spaces $\mathcal{E}(\varphi_n(x))$ and $\mathcal{F}(\varphi_n(x))$ to $\mathbb{R}^{d^\varepsilon} \times \{0\}$ and $\{0\} \times \mathbb{R}^{d^\mathcal{F}}$. Since the isomorphisms are bounded we get a sequence of diffeomorphisms as in Definition 3.21. Theorem 3.22 provides a plaque in \mathcal{N}_x and which depends continuously on x .

Theorem 3.20 is proved similarly: let \bar{u} be a half generalized orbit and $(y(t))_{t \in [0, \infty)}$ be its projection to K and \bar{P} the generalized flow; we consider the local diffeomorphisms $\bar{P}_1: \mathcal{N}_{y(n)} \rightarrow \mathcal{N}_{y(n+1)}$ in a neighborhood of the points $u(n)$ and $u(n+1)$ respectively, for each $n \in \mathbb{N}$.

3.3.5 Uniqueness

There is no uniqueness in Theorem 3.22, but once we have fixed the way of choosing \widehat{f}_n , the invariant graph becomes unique (and this is used to prove the continuity in Theorem 3.22). Also the following classical lemma holds.

Proposition 3.23. *In the setting of Theorem 3.22, up to reduce α , the following property holds. If there exists z' in the graph of $\psi(\underline{F})$ and $z \in \mathbb{R}^d$ such that for any $n \geq 0$*

- the iterates of z and z' by $f_n \circ \dots \circ f_0$ are defined and belong to $B(0, \alpha)$,
- $(f_n \circ \dots \circ f_0(z)) - (f_n \circ \dots \circ f_0(z')) \in \mathcal{C}_1$,

then z is also contained in the graph of $\psi(\underline{F})$.

Proof. Let us assume by contradiction that $z = (x, y)$ is not contained in the graph of ψ and let us denote $\widehat{z} = (x, \psi(x))$. The line containing the iterates of z and \widehat{z} by the sequence $f_{n-1} \circ \dots \circ f_0$, $n \geq 1$, remains tangent to the cone $\mathbb{R}^d \setminus \mathcal{C}_1$ (by the dominated splitting). The line containing the iterates of z and z' and the line containing the iterates of \widehat{z} and z' are tangent to \mathcal{C}_1 by our assumption, and by the two items of Theorem 3.22.

The domination implies that the distances between the n^{th} iterates of z, \widehat{z} gets exponentially larger than their distance to the n^{th} iterate of z' . This contradicts the triangular inequality. \square

Remark 3.24. The plaque family \mathcal{W}^{cs} given by Theorem 3.19 is a priori only invariant by the time-1 map P_1 of the flow but the previous proposition shows that if $\alpha_\varepsilon > 0$ is small enough, for any $x \in K$ and $z \in \mathcal{W}^{cs}(x)$ such that $P_n(z) \in \mathcal{W}_{\alpha_\varepsilon}^{cs}(\varphi_n(x))$ for each $n \in \mathbb{N}$, then we have $P_t(z) \in \mathcal{W}^{cs}(\varphi_t(x))$ for any $t > 0$. Indeed, by invariance of the cone \mathcal{C}^F , the point $P_t(z)$ belongs to $\mathcal{C}^\varepsilon(\varphi_t(x))$ for any $t \geq 0$.

The same property holds for half generalized orbits parametrized by $[0, +\infty)$.

3.3.6 Coherence

The uniqueness allows us to deduce that when plaques intersect then they have to *match*, i.e. to be contained in a larger sub-manifold.

Proposition 3.25. Fix a plaque family \mathcal{W}^{cs} as given by Theorem 3.20. Up to reduce the constants $\eta, \alpha_\varepsilon > 0$ the following property holds. Let us consider any half generalized orbits \bar{u}, \bar{u}' (parametrized by $[0, +\infty)$), and any sets $X \subset \mathcal{W}_{\alpha_\varepsilon}^{cs}(\bar{u})$, $X' \subset \mathcal{W}_{\alpha_\varepsilon}^{cs}(\bar{u}')$ such that:

1. \bar{u}, \bar{u}' belong to the η -neighborhood of K and satisfy $\bar{u} \in X$, $\bar{u}' \in X'$,
2. the points y, y' satisfying $u(0) \in \mathcal{N}_y$, $u'(0) \in \mathcal{N}_{y'}$ are r_0 close and y belongs to the $2r_0$ -neighborhood of $K \setminus V$,
3. the projection $(y(t))_{t \in [0, +\infty)}$ of \bar{u} to K has arbitrarily large iterates in the r_0 -neighborhood of $K \setminus V$,
4. $\pi_y(X') \cap X \neq \emptyset$ and $\text{Diam}(\bar{P}_t(X)), \text{Diam}(\bar{P}_t(X')) \leq \alpha_\varepsilon$ for all $t \geq 0$,

then $\pi_y(X')$ is contained in $\mathcal{W}^{cs}(\bar{u})$.

Proof. Let us denote by $(y(t))$ and $(y'(t))$ the projections of \bar{u} and \bar{u}' to K . Proposition 3.23 gives $\alpha > 0$. Provided η, α_ε are small enough, the proof consists in checking that some Global invariance extends to generalized orbits:

Claim. There exist $T, T' > 10$ such that

- (a) the points $y(T), y'(T')$ are r_0 close and $y(T)$ belongs to the $2r_0$ -neighborhood of $K \setminus V$,
- (b) $\bar{P}_T(\pi_y(X')) = \pi_{y(T)}(\bar{P}_{T'}(X'))$,
- (c) $\text{Diam}(\bar{P}_t(\pi_y(X'))) \leq \alpha/2$ for any $t \in [0, T]$.

Proof of the Claim. Let C be a large constant which bounds:

- the Lipschitz constant of the projections π_z , for z in the $2r_0$ -neighborhood of $K \setminus V$,
- the norms $\|DP_s\|$ for $|s| \leq 1$.

Since φ is continuous, there exists $\tilde{\eta} > \eta$ such that for each $z, z' \in K$ satisfying $d(z, z') < \eta$ we have $\sup_{s \in [-1, 1]} d(\varphi_s(z), \varphi_s(z')) < \tilde{\eta}$. Taking η small allows to choose $\tilde{\eta}$ small as well.

We will apply a first time the Global invariance, with the constants $\rho = 2$ and $\delta_1 := \min(r_0/4, \alpha/4)$: it gives us constants β_1, r_1 . Then we will apply a second time the Global invariance (the version of Remark 3.5.(e)), with the constant $\rho = 2$ and $\delta_2 := r_1/4$: it gives us constant β_2 . Take η and α_ε small so that $(1+C)^2(\tilde{\eta} + \alpha_\varepsilon) < \min(\beta_1, \beta_2)$ and $\eta < r_0/4$, $\tilde{\eta} < r_1/2$.

For proving the claim, it is enough to prove the existence of $T, T' > 0$ satisfying the properties (a), (b), (c) above and such that one of the following properties occurs:

- $T, T' > 10$,
- $[0, T]$ contains a discontinuity of $(y(t))$,
- $[0, T']$ contains a discontinuity of $(y'(t))$.

Indeed by definition the discontinuities of generalized orbits are separated in time by at least 1. It is thus enough to apply the argument below 20 times in order to get a pair of times (T, T') such that $T, T' > 10$.

We now explain how to obtain the pair (T, T') . The diameters of $\{0_y\} \cup X$ and of $\{0_{y'}\} \cup X'$ are smaller than $\eta + \alpha_\varepsilon$. Moreover $\pi_y(X')$ meets X . Hence $\|\pi_y(y')\| < (1 + C)(\eta + \alpha_\varepsilon)$. With our choice of η, α_ε , this gives $\|\pi_y(y')\| < C^{-1} \min(\beta_1, \beta_2)$ and then $\|P_s \circ \pi_y(y')\| < \beta_2$ for any $s \in [-1, 1]$. The Global invariance (Remark 3.5.(e)) gives $\theta_2 \in \text{Lip}_{1+\rho}$ such that $|\theta_2(0)| \leq 1/4$, and $d(\varphi_s(y), \varphi_{\theta_2(s)}(y)) < \delta_2$ for any $s \in [-1, 1]$.

First case: $\theta_2(0) \geq 0$. We estimate

$$d(y, y'(\theta_2(0))) \leq d(y, \varphi_{\theta_2(0)}(y')) + d(\varphi_{\theta_2(0)}(y'), y'(\theta_2(0))).$$

We have $d(y, \varphi_{\theta_2(0)}(y')) = d(\varphi_0(y), \varphi_{\theta_2(0)}(y')) < \delta_2$. Note that either $\varphi_{\theta_2(0)}(y') = y'(\theta_2(0))$, or, the generalized orbit \bar{u}' has one discontinuity at some time s which belongs to $[0, \theta_2(0)] \subset [-1, 1]$. Since \bar{u}' is in the η -neighborhood of K , we have $d(\varphi_s(y'), y'(s)) < \eta$. One thus gets that $d(\varphi_{\theta_2(0)}(y'), y'(\theta_2(0))) < \tilde{\eta}$ and $d(y, y'(\theta_2(0))) < \delta_2 + \tilde{\eta} < r_1$.

We can thus apply the Global invariance to the points $y \in U$ and $y'(\theta_2(0))$ and to points v in X , $v' \in X'$ such that $\pi_y(v') = v$. Using the local invariance and a previous estimate we get: $\|\pi_y(y'(\theta_2(0)))\| = \|\pi_y(y')\| < \beta_1$. Consequently, there exists $\theta_1 \in \text{Lip}_{1+\rho}$ and an interval $[0, a)$ such that $d(\varphi_t(y), \varphi_{\theta_1(t)}(y'(\theta_2(0)))) < \delta_1 = r_0$ for any $t \in [0, a)$. Here $[0, a)$ is any interval such that $\|P_t(v)\| < \beta_1$ and $\|P_{\theta_1(t)+\theta_2(0)}(v')\| < \beta_1$. Since \bar{u}, \bar{u}' are in the η -neighborhood of K and by the assumption (4) above, this is ensured if $[0, a)$ is the maximal interval of time t such that $\varphi_t(y) = y(t)$ and $\varphi_{\theta_1(t)}(y'(\theta_2(0))) = y'(\theta_2(0) + \theta_1(t))$.

If $a < +\infty$, we set $T = a$ and $T' = \theta_2(0) + \theta_1(a)$. By definition of generalized orbits, either $y(T)$ or $y'(T')$ is in the η -neighborhood of $K \setminus V$ and they are at distance smaller than $\delta_1 + 2\eta < r_0$. In particular both $y(T)$ and $y'(T')$ belong to the $2r_0$ -neighborhood of $K \setminus V$. Using again the condition $\|u(t)\| + \text{Diam}(\bar{P}_t(X')) < \eta + \alpha_\varepsilon < \min(\beta_1, \beta_2)$, the Global invariance gives the condition (b) and $\|\bar{P}_t(w)\| \leq \delta_1 = \alpha/4$ for each $t \in [0, T]$ and each $w \in \pi_y(X')$. The conditions on T, T' are thus satisfied.

If $a = \infty$, we use the assumption (3) to find a large time T such that $y(T)$ belongs to the r_0 -neighborhood of $K \setminus V$ and we set $T' = \theta_2(0) + \theta_1(a)$. The conditions on (T, T') are checked similarly (this case is simpler).

Second case: $\theta_2(0) < 0$. We follow the argument of the first case. As above, $d(y', y(\theta_2^{-1}(0))) < r_1$ and we apply the Global invariance to the points $y' \in U$ and $y(\theta_2^{-1}(0))$. This gives $\theta_1 \in \text{Lip}_{1+\rho}$. We choose $T' > 0$ and set $T = \theta_2^{-1}(0) + \theta_1^{-1}(T')$ such that either $y(T)$ or $y'(T')$ is a discontinuity of the family $(y(t))$ or $(y'(t'))$, or $y(T) \in K \setminus V$. \square

Applying the Claim inductively, we find two increasing sequences of times $T_n, T'_n \rightarrow +\infty$. Indeed having defined T_n, T'_n , the generalized orbits $\bar{P}_{T_n}(\bar{u})$ and $\bar{P}_{T'_n}(\bar{u}')$ satisfy the assumptions of Proposition 3.25 and the Claim associate a pair T, T' ; we then set $T_{n+1} = T_n + T$ and $T'_{n+1} = T'_n + T'$.

In order to conclude Proposition 3.25, one considers $z' \in \pi_y(X') \cap X$ and any $z \in \pi_y(X')$ and use Proposition 3.23. Let us check that its assumptions are satisfied:

- By our assumptions, and requiring $\alpha_{\mathcal{E}} < \alpha$ we have $\|\bar{P}_t(z') - u(t)\| < \alpha/2$ for any $t \geq 0$.
- Since $\bar{P}_t(z'), \bar{P}_t(z) \in \bar{P}_t(\pi_y(X'))$, the first item of the lemma implies that we also have $\|\bar{P}_t(z) - u(t)\| \leq \alpha$ for any $t \geq 0$.
- Since the complement of the cone field $\mathcal{C}^{\mathcal{E}}$ is invariant by forward iterations, it only remains to check that $\bar{P}_t(z) - \bar{P}_t(z') \in \mathcal{C}^{\mathcal{E}}(y(t))$ for a sequence of arbitrarily large times t .

From the second item of the lemma, the projections of the points $\bar{P}_{T_n}(z), \bar{P}_{T_n}(z')$ by $\pi_{y'(T'_n)}$ belong to $\bar{P}_{T'_n}(\mathcal{W}^{cs}(\bar{u}'))$, and hence to $\mathcal{W}^{cs}(\bar{P}_{T'_n}(\bar{u}'))$ by Remark 3.24. So their difference belongs to $\mathcal{C}_{1/2}^{\mathcal{E}}(y'(T'_n))$ (by Theorem 3.22). The continuity of the cone field and the fact $d(y(T_n), y'(T'_n)) < \delta_0$ gives $\bar{P}_{T_n}(z) - \bar{P}_{T_n}(z') \in \mathcal{C}^{\mathcal{E}}(y(T_n))$.

Hence Proposition 3.23 applies and concludes the proof of Proposition 3.25. \square

3.3.7 Limit dynamics in periodic fibers

We state a consequence of the existence of plaque families. It will be used for the center-unstable plaques \mathcal{W}^{cu} . Note that any $u \in \mathcal{N}$ such that $\|P_{-t}(u)\|$ is small for any $t \geq 0$ is a half generalized orbit, hence has a plaque $\mathcal{W}^{cu}(u)$.

Proposition 3.26. *For any local fibered flow (P_t) on a bundle \mathcal{N} admitting a dominated splitting $\mathcal{N} = \mathcal{E} \oplus \mathcal{F}$ where \mathcal{E} is one-dimensional, there exists $\delta > 0$ with the following property.*

For any periodic point $z \in K$ with period T and any $u \in \mathcal{N}_z$ satisfying $\|P_{-t}(u)\| \leq \delta$ for all $t > 0$ and $0_z \notin \mathcal{W}^{cu}(u)$, there exists $p \in \mathcal{N}_z$ such that $P_{2T}(p) = p$ and $P_{-t}(u)$ converges to the orbit of p when t goes to $+\infty$.

Proof. Let $\alpha_{\mathcal{E}}, \alpha_{\mathcal{F}}$ be the constants associated to \mathcal{E}, \mathcal{F} as in Theorem 3.20. The plaques $\mathcal{W}^{cu}(u)$ and $\mathcal{W}^{cs}(0_z)$ intersect at a (unique) point y .

Since $\|u\|$ is small, by the local invariance of the plaque families, y is also the intersection between $P_1(\mathcal{W}_{\alpha_{\mathcal{E}}}^{cs}(\varphi_{-1}(z)))$ and $\mathcal{W}_{\alpha_{\mathcal{F}}}^{cu}(u)$. One deduces that $P_{-1}(y)$ is the (unique) intersection point between the plaques $\mathcal{W}^{cu}(P_{-1}(u))$ and $\mathcal{W}^{cs}(\varphi_{-1}(z))$. Repeating this argument inductively, one deduces that the backward orbit of y by P_{-1} remain in the plaques $\mathcal{W}^{cu}(P_{-k}(u))$ and $\mathcal{W}^{cs}(\varphi_{-k}(z))$. From Remark 3.24, any backward iterate $P_{-t}(y)$ belongs to $\mathcal{W}^{cu}(P_{-t}(u))$.

Since $y \neq 0_z$, the domination implies that $d(P_{-k}(u), P_{-k}(y))$ is exponentially smaller than $d(P_{-k}(y), 0_{\varphi_{-k}(z)})$ as $k \rightarrow +\infty$. So $d(P_{-k}(u), P_{-k}(y))$ goes to 0. The same argument applied to $P_{-s}(u)$, $s \in [0, 1]$ shows that the distance of $P_{-t}(u)$ to $\mathcal{W}^{cs}(\varphi_{-t}(z))$ converges to 0 as $t \rightarrow +\infty$. In particular, the limit set of the orbit of u under P_{-2T} is a closed subset L of $\mathcal{W}^{cs}(z)$. In the case L is a single point p , the conclusion of the proposition follows.

We assume now by contradiction that L is not a single point. There exists $q \neq 0_z$ invariant by P_{2T} in L such that L intersects the open arc γ in $\mathcal{W}^{cs}(z)$ bounded by 0_z and q . Note that the forward iterates $P_k(\gamma)$ by P_1 remain small, hence in $\mathcal{W}^{cs}(\varphi_k(z))$ by the local invariance of \mathcal{W}^{cs} . From Remark 3.24, any iterate $P_t(\gamma)$ is contained in $\mathcal{W}^{cs}(\varphi_t(z))$, $t \in \mathbb{R}$.

Up to replace u by a backward iterate, one can assume that y belongs to γ . This shows that $P_{-t}(y)$ is the intersection between $\mathcal{W}^{cu}(P_{-t}(u))$ and $\mathcal{W}^{cs}(P_{-t}(z))$ for any $t \geq 0$. The set L is thus the limit set of the orbit of y under P_{-2T} . This reduces to a one-dimensional dynamics for an orientation preserving diffeomorphism, and L has to be a single point, a contradiction. \square

3.3.8 Distortion control

The following lemma restates the classical Denjoy-Schwartz argument in our setting.

Lemma 3.27. *Let us assume that (P_t) is C^2 , that \mathcal{E} is one-dimensional and that \mathcal{W}^{cs} is a C^2 locally invariant plaque family. Then, there is $\beta_S > 0$ and for any $C_{Sum} > 0$, there are $C_S, \eta_S > 0$ with the following property.*

For any $x \in K$, for any interval $I \subset \mathcal{W}^{cs}(x)$ and any $n \in \mathbb{N}$ satisfying

$$\forall m \in \{0, \dots, n\}, P_m(I) \subset B(0, \beta_S) \text{ and } \sum_{m=0}^n |P_m(I)| \leq C_{Sum},$$

then (1) for any $u, v \in I$ we have

$$C_S^{-1} \leq \frac{\|DP_n(u)|_I\|}{\|DP_n(v)|_I\|} \leq C_S; \quad (8)$$

in particular $\|DP_n(u)|_I\| \leq C_S \frac{|P_n(I)|}{|I|}$;

(2) any interval $\hat{I} \subset \mathcal{W}^{cs}(x)$ containing I with $|\hat{I}| \leq (1 + \eta_S)|I|$ satisfies $|P_n(\hat{I})| \leq 2|P_n(I)|$;

(3) any interval $\hat{I} \subset \mathcal{W}^{cs}(x)$ containing I with $|P_n(\hat{I})| \leq (1 + \eta_S)|P_n(I)|$ satisfies $|\hat{I}| \leq 2|I|$.

The proof is similar to [dMvS, Chapter I.2].

3.4 Hyperbolic iterates

We continue with the setting of Section 3.3 and we fix two locally invariant plaques families \mathcal{W}^{cs} and \mathcal{W}^{cu} tangent to \mathcal{E} and \mathcal{F} respectively, and two constants $\alpha_{\mathcal{E}}, \alpha_{\mathcal{F}}$ controlling the geometry and the dynamics inside these plaques as in the previous sections. The plaques are defined at points of K but also at half generalized orbits \bar{u} contained in a η -neighborhood of K and parametrized by $[0, +\infty)$ and $(-\infty, 0]$ respectively. The quantities $\eta, \alpha_{\mathcal{E}}, \alpha_{\mathcal{F}} > 0$ may be reduced in order to satisfy further properties below.

In case \mathcal{E} is 2-dominated, \mathcal{W}^{cs} will be a C^2 -plaque family. Remember that τ_0, λ are the constants associated to the domination, as introduced in Section 3.3.1.

3.4.1 Hyperbolic points

We introduce a first notion of hyperbolicity.

Definition 3.28. *Let us fix $C_{\mathcal{E}}, \lambda_{\mathcal{E}} > 1$. A piece of orbit $(x, \varphi_t(x))$ in K is $(C_{\mathcal{E}}, \lambda_{\mathcal{E}})$ -hyperbolic for \mathcal{E} if for any $s \in (0, t)$, we have*

$$\|DP_s|_{\mathcal{E}(x)}\| \leq C_{\mathcal{E}} \lambda_{\mathcal{E}}^{-s}.$$

A point x is $(C_{\mathcal{E}}, \lambda_{\mathcal{E}})$ -hyperbolic for \mathcal{E} if $(x, \varphi_t(x))$ is $(C_{\mathcal{E}}, \lambda_{\mathcal{E}})$ -hyperbolic for \mathcal{E} for any $t > 0$. We have similar definitions for the bundle \mathcal{F} (considering the flow $t \mapsto P_{-t}$).

By the continuity of the fibered flow for the C^1 -topology, the hyperbolicity extends to orbits close (the proof is easy and omitted).

Lemma 3.29. *Let us assume that \mathcal{E} is one-dimensional. For any $\lambda' > 1$, there exist $C', \delta, \rho > 0$ such that for any $x, y \in K$, $t > 0$ and $\theta \in \text{Lip}_{1+\rho}$ satisfying $\theta(0) = 0$ and*

$$d(\varphi_s(x), \varphi_{\theta(s)}(y)) < \delta \text{ for each } s \in [0, t],$$

then

$$\|DP_{\theta(t)}|_{\mathcal{E}(y)}\| \leq C' \lambda'^t \|DP_t|_{\mathcal{E}(x)}\|.$$

Hyperbolicity implies summability for the iterations inside one-dimensional plaques.

Lemma 3.30 (Summability). *Let us assume that \mathcal{E} is one-dimensional and consider $\lambda_{\mathcal{E}}, C_{\mathcal{E}} > 1$. Then, there exists $C'_{\mathcal{E}} > 1$ and $\delta_{\mathcal{E}} > 0$ with the following property.*

For any piece of orbit $(x, \varphi_t(x))$ which is $(C_{\mathcal{E}}, \lambda_{\mathcal{E}})$ -hyperbolic for \mathcal{E} for any interval $I \subset \mathcal{W}^{cs}(x)$ containing 0 whose length $|I|$ is smaller than $\delta_{\mathcal{E}}$, and for any interval $J \subset I$ one has

$$|P_t(J)| \leq C_{\mathcal{E}} \lambda_{\mathcal{E}}^{-t/2} |J| \text{ and } \sum_{0 \leq m \leq [t]} |P_m(J)| \leq C'_{\mathcal{E}} |J|.$$

Proof. Let $\eta > 0$ be small such that $1 + \eta < \lambda_{\mathcal{E}}^{1/2}$. Then, there exists δ_0 such that for any $y \in K$ and any interval $I_0 \subset \mathcal{W}^{cs}(y)$ containing 0 whose length is smaller than δ_0 , one has

$$\forall s \in [0, 1], \quad |P_s(I_0)| \leq (1 + \eta) \|DP_s|_{\mathcal{E}(y)}\| |I_0|.$$

Let us choose $\delta_{\mathcal{E}}$ satisfying $\delta_{\mathcal{E}} C_{\mathcal{E}} \lambda_{\mathcal{E}}^{1/2} < \delta_0$. One checks inductively that the length of $P_k(I)$ is smaller than δ_0 for each $k \in [0, t]$. The conclusion of the lemma follows. \square

3.4.2 Pliss points

We introduce a more combinatorial notion of hyperbolicity (only used for \mathcal{F}).

Definition 3.31. *For $T \geq 0$ and $\gamma > 1$, we say that a piece of orbit $(\varphi_{-t}(x), x)$ is a (T, γ) -Pliss string (for the bundle \mathcal{F}) if there exists an integer $s \in [0, T]$ such that*

$$\text{for any integer } m \in \left[0, \frac{t-s}{\tau_0}\right], \quad \prod_{n=0}^{m-1} \|DP_{-\tau_0}|_{\mathcal{F}(\varphi_{-(n\tau_0+s)}(x))}\| \leq \gamma^{-m\tau_0}.$$

A point x is (T, γ) -Pliss (for \mathcal{F}) if $(\varphi_{-t}(x), x)$ is a (T, γ) -Pliss string for any $t > 0$.

For simplicity, a piece of orbit $(\varphi_{-t}(x), x)$ is a T -Pliss string if it is a (T, λ) -Pliss string and x is T -Pliss if it is (T, λ) -Pliss, where λ is the constant for the domination.

For any $T > 0$, there exists $C > 1$ such that any piece of orbit which is a T -Pliss string is also (C, λ) -hyperbolic for \mathcal{F} . Pliss lemma gives a kind of converse result.

Lemma 3.32. *Assume $\dim \mathcal{F} = 1$. For any $\lambda_1 > \lambda_2 > 1$ and $C > 1$, there is $T_{\mathcal{F}} > 0$ such that any piece of orbit $(\varphi_{-t}(x), x)$ which is (C, λ_1) -hyperbolic for \mathcal{F} is also a $(T_{\mathcal{F}}, \lambda_2)$ -Pliss string.*

Proof. Let us define $a_i = \log \|DP_{-\tau_0}|_{\mathcal{F}(\varphi_{-i\tau_0}(x))}\|$ for $0 \leq i \leq \frac{t}{\tau_0} - 1$. Then by applying a usual Pliss lemma, one can get the conclusion (see for instance [MCY, Lemma 2.3]). \square

Pliss strings extend to orbits close.

Lemma 3.33. *If \mathcal{F} is one-dimensional, there exist $\beta > 0$ and $T \geq 1$ with the following property: if $(x, \varphi_t(x))$ is a $T_{\mathcal{F}}$ -Pliss string with $x \in U$ and $T_{\mathcal{F}} \geq T$ and if y satisfies:*

$$d(y, x) < r_0 \text{ and for any } 0 \leq s \leq t \text{ one has } \|P_s \pi_x(y)\| < \beta,$$

then there exists a homeomorphism $\theta \in \text{Lip}_2$ such that

- $|\theta(0)| \leq 1/4$ and $d(\varphi_s(x), \varphi_{\theta(s)}(y)) < r_0/2$ for each $s \in [-1, t+1]$,
- $\pi_{\varphi_s(x)} \circ \varphi_{\theta(s)}(y) = P_s \circ \pi_x(y)$ when $s \in [0, t]$ and $\varphi_s(x) \in U$,
- the piece of orbit $(y, \varphi_{\theta(t)+a}(y))$ is a $(2T_{\mathcal{F}}, \lambda^{1/2})$ -Pliss string for any $a \in [-1, 1]$.

Proof. The continuity of the flow for the C^1 -topology implies that there are $\delta > 0$ and $\rho \in (0, 1/20)$ such that for any $z, z' \in K$ and any $\theta \in \text{Lip}_{1+\rho}$, if $d(\varphi_s(z), \varphi_{\theta(s)}(z')) < \delta$ for any $0 \leq s \leq \tau_0$, then

$$\|DP_{\theta(0)-\theta(\tau_0)}|F(\varphi_{\theta(\tau_0)}(z'))\| \leq \lambda^{\tau_0/5} \|DP_{-\tau_0}|F(z')\|.$$

The Global invariance associates to $\delta, \rho > 0$ some constants $r, \beta > 0$.

Consider $z, z' \in K$, $n \geq 0$ and $\theta \in \text{Lip}_{1+\rho}$ such that $(z, \varphi_{n\tau_0}(z))$ is 0-Pliss and

$$d(\varphi_s(z), \varphi_{\theta(s)}(z')) < \delta, \quad \forall 0 \leq s \leq n\tau_0.$$

By the choice of δ, ρ , we have for any $0 \leq k \leq n-1$,

$$\prod_{j=k+1}^n \|DP_{\theta((j-1)\tau_0)-\theta(j\tau_0)}|F(\varphi_{\theta(j\tau_0)}(z'))\| < \lambda^{-4/5(n-k)\tau_0} < \lambda^{-3/4[\theta(n\tau_0)-\theta(k\tau_0)]}.$$

Thus there is $C > 0$ depending on λ and $\sup_{s \in [0, 2\tau_0]} \|DP_{-s}\|$ such that

$$\|DP_{-s}|F(\varphi_{\theta(n\tau_0)-d}(z'))\| \leq C\lambda^{-3/4s}, \quad \forall s \in [0, \theta(n\tau_0) - d + \tau_0], \quad d \in [0, 1].$$

Now, by our assumptions and the Global invariance, there is $\theta \in \text{Lip}_{1+\rho}$ such that

$$d(\varphi_s(x), \varphi_{\theta(s)}(y)) < \delta, \quad \forall s \in [-1, t+1].$$

Consider T associated to $\lambda^{3/4}, \lambda^{1/2}, C$ by Lemma 3.32. For any $T_{\mathcal{F}} > 2T$, the fact that $(x, \varphi_t(x))$ is $(T_{\mathcal{F}}, \lambda)$ -Pliss implies that there is an integer $b \in [0, T_{\mathcal{F}}]$ such that $(x, \varphi_{t-b}(x))$ is 0-Pliss. One chooses $d \in [0, 1]$ such that $\theta(t) + a - \theta(t-b) + d$ is a nonnegative integer.

One deduces that for any $s \in [0, \theta(t-b) - d]$,

$$\|DP_{-s}|F(\varphi_{\theta(t-b)-d}(z))\| \leq C\lambda^{-3/4s}.$$

By Lemma 3.32, there is an integer $e \in [0, T]$ such that $(z, \varphi_{\theta(t-b)-d-e}(z))$ is $(0, \lambda^{1/2})$ -Pliss. Now $\theta(t-b) - d - e$ differs from $\theta(t) + a$ by an integer smaller than $(1+\rho)T_{\mathcal{F}} + 2 + T$, which is smaller than $2T_{\mathcal{F}}$. Hence $(z, \varphi_{\theta(t)}(z))$ is $(2T_{\mathcal{F}}, \lambda^{1/2})$ -Pliss for \mathcal{F} . \square

The next proposition will allow us to find iterates that are Pliss points and belong to U .

Proposition 3.34. *Let us assume that \mathcal{E} is one-dimensional and let W be a set such that \mathcal{E} is uniformly contracted on $\bigcup_{s \in [0, 1]} \varphi_s(W)$. Then, there exist $C_{\mathcal{E}}, \lambda_{\mathcal{E}} > 1$ such that any $T_{\mathcal{F}} \geq 0$ large enough has the following property.*

If there are $x \in K$ and integers $0 \leq k \leq \ell$ such that

- $(\varphi_{-\ell}(x), \varphi_{-k}(x))$ is a $T_{\mathcal{F}}$ -Pliss string,
- for any $j \in \{1, \dots, k-2\}$ either $\varphi_{-j}(x) \in W$ or the piece of orbit $(\varphi_{-\ell}(x), \varphi_{-j}(x))$ is not a $T_{\mathcal{F}}$ -Pliss string,

then $(\varphi_{-k}(x), x)$ is $(C_{\mathcal{E}}, \lambda_{\mathcal{E}})$ -hyperbolic for \mathcal{E} .

Similarly if there are $x \in K$ and $k \geq 0$ such that

- $\varphi_{-k}(x)$ is $T_{\mathcal{F}}$ -Pliss,
- for any $j \in \{1, \dots, k-2\}$ either the point $\varphi_{-j}(x)$ is in W or $\varphi_{-j}(x)$ is not $T_{\mathcal{F}}$ -Pliss,

then $(\varphi_{-k}(x), x)$ is $(C_{\mathcal{E}}, \lambda_{\mathcal{E}})$ -hyperbolic for \mathcal{E} .

Proof. The proof is essentially contained in [CP, Lemma 9.20]. Recall that $\lambda > 1$ is the constant for the domination. There exist $C_0, \lambda_0 > 1$ such that for any piece of orbit $(y, \varphi_t(y))$ in $\bigcup_{s \in [0,1]} \varphi_s(W)$, one has $\|DP_t|_{\mathcal{E}(y)}\| \leq C_0 \lambda_0^{-t}$. Let $C_1 > 1$ such that

$$\forall y \in K, \forall s \in [-\tau_0, \tau_0], \quad \|DP_s|_{\mathcal{E}(y)}\| \leq C_1 \lambda^{-s}. \quad (9)$$

One takes $C_{\mathcal{E}} = C_0^2 C_1^3$ and $\lambda_{\mathcal{E}} > 1$ smaller than $\min(\lambda, \lambda_0)$. One then chooses $T_{\mathcal{F}} \geq 0$ large such that $C_0 C_1 \lambda^{-T_{\mathcal{F}}} < \lambda_{\mathcal{E}}^{-T_{\mathcal{F}}}$.

Let $x \in K$, $0 \leq k \leq \ell$ be as in the statement of the lemma. We introduce the set

$$\mathcal{P} = \left\{ j \in \{1, \dots, k-2\}, (\varphi_{-\ell}(x), \varphi_{-j}(x)) \text{ is a } T_{\mathcal{F}}\text{-Pliss string} \right\}.$$

The set \mathcal{P} decomposes into intervals $\{a_i, 1 + a_i, \dots, b_i\} \subset \{1, \dots, k-2\}$, with $i = 1, \dots, i_0$, such that $b_i + 1 < a_{i+1}$. By convention we set $b_0 = 0$.

Claim 1. $b_i - a_i \geq T_{\mathcal{F}}$ unless $\{a_i, \dots, b_i\}$ contains 1 or $k-2$.

Proof. By maximality of each interval, $(\varphi_{-\ell}(x), \varphi_{-b_i}(x))$ has to be a 0-Pliss string. \square

Claim 2. Consider $n_1, n_2 \in \{0, \dots, k\}$ with $n_1 < n_2$ and such that $(\varphi_{-\ell}(x), \varphi_{-n_2}(x))$ is a 0-Pliss string and $(\varphi_{-\ell}(x), \varphi_{-j}(x))$ is not a 0-Pliss string for $n_1 < j < n_2$. Then for any $0 \leq m < (n_2 - n_1)/\tau_0$,

$$\|DP_{m\tau_0}|_{\mathcal{E}(\varphi_{-n_2}(x))}\| \leq \lambda^{-m\tau_0}.$$

Proof. One checks inductively that

$$\prod_{n=0}^{m-1} \|DP_{\tau_0}|_{\mathcal{F}(\varphi_{-n_2+n\tau_0}(x))}\| \leq \lambda^{m\tau_0}. \quad (10)$$

Indeed if this inequality holds up to an integer $m-1$ and fails for m , the piece of orbit $(\varphi_{-n_2}(x), \varphi_{-n_2+m\tau_0}(x))$ is a 0-Pliss string. It may be concatenate with $(\varphi_{-\ell}(x), \varphi_{-n_2}(x))$, implying that $(\varphi_{-\ell}(x), \varphi_{-n_2+m\tau_0}(x))$ is a 0-Pliss string. This is a contradiction since $n_2 - m\tau_0 > n_1$.

The estimate of the claim follows from (10) by domination. \square

The proposition will be a consequence of the following properties.

Claim 3. 1. The piece of orbit $(\varphi_{-b_i}(x), \varphi_{-b_{i-1}}(x))$ is $(C_0 C_1, \lambda_{\mathcal{E}})$ -hyperbolic for \mathcal{E} , for any $i \in \{1, \dots, i_0\}$, unless $i = i_0$ and $b_{i_0} = k-2$.

Moreover if $i \neq 1$, then $\|DP_{b_i-b_{i-1}}|_{\mathcal{E}(\varphi_{-b_i}(x))}\| \leq \lambda_{\mathcal{E}}^{-(b_i-b_{i-1})}$.

2. If $b_{i_0} = k-2$, then $(\varphi_{-k}(x), \varphi_{-b_{i_0-1}}(x))$ is $(C_0 C_1, \lambda_{\mathcal{E}})$ -hyperbolic for \mathcal{E} .

3. If $b_{i_0} < k-2$, then $(\varphi_{-k}(x), \varphi_{-b_{i_0}}(x))$ is $(C_0 C_1, \lambda_{\mathcal{E}})$ -hyperbolic for \mathcal{E} .

Proof. In order to check the first item, one introduces the smallest $j \in \{a_i, \dots, b_i\}$ such that $(\varphi_{-\ell}(x), \varphi_{-j}(x))$ is a 0-Pliss string: it exists unless $i = i_0$ and $b_{i_0} = k-2$. By our assumptions, the piece of orbit $(\varphi_{-b_i}(x), \varphi_{-j}(x))$ is contained in $\bigcup_{s \in [0,1]} \varphi_s(W)$, hence is $(C_0, \lambda_{\mathcal{E}})$ -hyperbolic for \mathcal{E} . Then Claim 2 gives $\|DP_{m\tau_0}|_{\mathcal{E}(\varphi_{-j}(x))}\| \leq \lambda^{-m\tau_0}$ for any $m \in \{0, \dots, (j-b_{i-1})/\tau_0\}$. One concludes the first part of item 1 by combining these estimates with (9).

Note that one also gets the estimate:

$$\|DP_{b_i-b_{i-1}}|_{\mathcal{E}(\varphi_{-b_i}(x))}\| \leq C_0 C_1 \lambda_{\mathcal{E}}^{-(b_i-b_{i-1})} \left(\frac{\lambda}{\lambda_{\mathcal{E}}} \right)^{-(j-b_{i-1})}.$$

If $i \geq 2$, one gets $j - b_{i-1} \geq j - a_i = T_{\mathcal{F}}$, hence by our choice of $T_{\mathcal{F}}$:

$$\|DP_{b_i-b_{i-1}}|\mathcal{E}(\varphi_{-b_i}(x))\| \leq C_0 C_1 \lambda_{\mathcal{E}}^{-(b_i-b_{i-1})} \left(\frac{\lambda}{\lambda_{\mathcal{E}}}\right)^{-T_{\mathcal{F}}} \leq \lambda_{\mathcal{E}}^{-(b_i-b_{i-1})}.$$

This gives the second part of item 1.

The proofs of items 2 and 3 are similar to the proof of item 1: for item 2, one introduces the smallest $j \geq a_{i_0}$ such that $(\varphi_{-\ell}(x), \varphi_{-j}(x))$ is a 0-Pliss string; for item 3, one introduces the smallest $j \geq k$ such that $(\varphi_{-\ell}(x), \varphi_{-j}(x))$ is a 0-Pliss string and use the fact that $(\varphi_{-\ell}(x), \varphi_{-k}(x))$ is a $T_{\mathcal{F}}$ -Pliss string. \square

From Claim 3, one first checks that for each $i \in \{1, \dots, i_0\}$

$$\|DP_{k-b_i}|\mathcal{E}(\varphi_{-k}(x))\| \leq C_0 C_1 \lambda_{\mathcal{E}}^{k-b_i}.$$

For each $m \in \{0, \dots, k\}$, either there exists $i \in \{1, \dots, i_0\}$ such that $b_{i-1} \leq m \leq b_i$ or $b_{i_0} \leq m \leq k$. Using items 1 or 3, one concludes

$$\|DP_{k-m}|\mathcal{E}(\varphi_{-k}(x))\| \leq C_0^2 C_1^2 \lambda_{\mathcal{E}}^{k-m}.$$

Combining with (9), one gets the required bound on $\|DP_{k-t}|\mathcal{E}(\varphi_{-k}(x))\|$ for any $t \in [0, k]$.

This proves the lemma for pieces of orbits $(\varphi_{-\ell}(x), x)$ such that $(\varphi_{-\ell}(x), \varphi_{-k}(x))$ is a $T_{\mathcal{F}}$ -Pliss string. The proof for half orbits $\{\varphi_{-t}(x), t > 0\}$ such that $\varphi_{-k}(x)$ is $T_{\mathcal{F}}$ -Pliss is similar. The proposition is proved. \square

3.4.3 Hyperbolic generalized orbits

The hyperbolicity extends to half generalized orbits. (Recall that if \bar{u} is parametrized by $(-\infty, 0]$, one can define the space $\mathcal{F}(\bar{u})$, see Subsection 3.3.3.)

Definition 3.35. *Let us fix $C_{\mathcal{F}}, \lambda_{\mathcal{F}} > 1$, $T_{\mathcal{F}} \geq 0$ and consider a half generalized orbit \bar{u} parametrized by $(-\infty, 0]$.*

\bar{u} is $(C_{\mathcal{F}}, \lambda_{\mathcal{F}})$ -hyperbolic for \mathcal{F} if for any $t \geq 0$, we have $\|D\bar{P}_t|\mathcal{F}(\bar{u})\| \leq C_{\mathcal{F}} \lambda_{\mathcal{F}}^{-t}$.

\bar{u} is $(T_{\mathcal{F}}, \lambda_{\mathcal{F}})$ -Pliss (for \mathcal{F}) if there exists an integer $s \in [0, T_{\mathcal{F}}]$ such that for any $m \in \mathbb{N}$,

$$\prod_{n=0}^{m-1} \|D\bar{P}_{-n\tau_0}|\mathcal{F}(\bar{P}_{-(n\tau_0+s)}(\bar{u}))\| \leq \lambda_{\mathcal{F}}^{-m\tau_0}.$$

One can define similarly hyperbolicity and Pliss property for pieces of generalized orbits. By continuity and invariance of the spaces $\mathcal{F}(\bar{u})$, one gets

Lemma 3.36. *For any $\lambda_{\mathcal{F}} \in (1, \lambda)$ and $T_{\mathcal{F}} \geq 0$, there exists $\eta > 0$ such that for any $y \in K$ which is $T_{\mathcal{F}}$ -Pliss and for any half generalized orbit \bar{u} parametrized by $(-\infty, 0]$, if*

$$\bar{u} \text{ is in the } \eta\text{-neighborhood of } K \text{ and its projection on } K \text{ is } (\varphi_{-t}(y))_{t \geq 0},$$

then \bar{u} is $(T_{\mathcal{F}}, \lambda_{\mathcal{F}})$ -Pliss.

Lemma 3.37. *For any $C_{\mathcal{F}}, \lambda_{\mathcal{F}} > 1$, there exists $\eta > 0$ such that for any piece of orbit $(\varphi_{-t}(y), y)$ which is $(C_{\mathcal{F}}/2, \lambda_{\mathcal{F}}^2)$ -hyperbolic for \mathcal{F} and for any half generalized orbit \bar{u} parametrized by $(-\infty, 0]$, if*

$$\bar{u} \text{ is in the } \eta\text{-neighborhood of } K \text{ and } u(-s) \in \mathcal{N}_{\varphi_{-s}(y)} \text{ for each } s \in (0, t),$$

then $(\bar{P}_{-t}(\bar{u}), \bar{u})$ is $(C_{\mathcal{F}}, \lambda_{\mathcal{F}})$ -hyperbolic for \mathcal{F} .

The proofs of Lemmas 3.36 and 3.37 are standard by continuity, hence are omitted.

3.4.4 Unstable manifolds

Pliss points have uniform unstable manifolds in the plaques (see e.g. [ABC, Section 8.2]).

Proposition 3.38. *Consider $\eta > 0$ and a center-unstable plaque family \mathcal{W}^{cu} as given by Theorem 3.20. For any $C_{\mathcal{F}}, \lambda_{\mathcal{F}} > 1$, $\beta_{\mathcal{F}} > 0$ and $T_{\mathcal{F}} \geq 0$, there exists $\alpha > 0$ such that for any half generalized orbit \bar{u} parametrized by $(-\infty, 0]$, in the η -neighborhood of K , if \bar{u} is $(T_{\mathcal{F}}, \lambda_{\mathcal{F}})$ -Pliss, or if \mathcal{F} is one-dimensional and \bar{u} is $(C_{\mathcal{F}}, \lambda_{\mathcal{F}})$ -hyperbolic for \mathcal{F} , then:*

$$\forall t \geq 0, \quad \text{Diam}(\bar{P}_{-t}(\mathcal{W}_{\alpha}^{cu}(\bar{u}))) \leq \beta_{\mathcal{F}} \lambda_{\mathcal{F}}^{-t/2}.$$

In particular, from Remark 3.24, the image $\bar{P}_{-t}(\mathcal{W}_{\alpha}^{cu}(\bar{u}))$ is contained in $\mathcal{W}^{cu}(\bar{P}_{-t}(\bar{u}))$.

3.4.5 Lipschitz holonomy and rectangle distortion

It is well-known that for a one-codimensional invariant foliation whose leaves are uniformly contracted, the holonomies between transversals are Lipschitz. In order to state a similar property in our setting we define the notion of rectangle.

Definition 3.39. A rectangle $R \subset \mathcal{N}_x$ is a subset which is homeomorphic to $[0, 1] \times B_{d-1}(0, 1)$ by a homeomorphism ψ , where $B_{d-1}(0, 1)$ is the $\dim(\mathcal{N}_x) - 1$ -dimensional unit ball such that:

- the set $\psi(\{0, 1\} \times B_{d-1}(0, 1))$ is a union of two C^1 -discs tangent to $\mathcal{C}^{\mathcal{F}}$, and is called the \mathcal{F} -boundary $\partial^{\mathcal{F}}R$,
- the curve $\psi([0, 1] \times \{0\})$ is C^1 and tangent to $\mathcal{C}^{\mathcal{E}}$.

A rectangle R has distortion bounded by $\Delta > 1$ if for any two C^1 -curves $\gamma, \gamma' \subset R$ tangent to $\mathcal{C}^{\mathcal{E}}$ with endpoints in the two connected components of $\partial^{\mathcal{F}}R$, then

$$\Delta^{-1}|\gamma| \leq |\gamma'| \leq \Delta|\gamma|.$$

Proposition 3.40. *Assume that the local fibered flow is C^2 . For any $C_{\mathcal{F}}, \lambda_{\mathcal{F}} > 1$, there exist $\Delta > 0$ and $\beta > 0$ with the following property. For any $y, \varphi_{-t}(y) \in K$ and $R \subset \mathcal{N}_y$ such that:*

- $(\varphi_{-t}(y), y)$ is $(C_{\mathcal{F}}, \lambda_{\mathcal{F}})$ -hyperbolic for \mathcal{F} ,
- $P_{-s}(R)$ is a rectangle and has diameter smaller than β for each $s \in [0, t]$,
- if D_1, D_2 are the two components of $\partial^{\mathcal{F}}(P_{-t}(R))$, then

$$d(D_1, D_2) > 10 \cdot \max(\text{Diam}(D_1), \text{Diam}(D_2)),$$

then the rectangle R has distortion bounded by Δ .

Proof. It is enough to prove the version of this result stated for a sequence of C^2 -diffeomorphisms with a dominated splitting. Then the argument is the same as [PS₁, Lemma 3.4.1]. \square

Remark 3.41. *In the previous statement, it is enough to replace the second condition by the weaker one: R is contained in $B(0_y, \beta)$.*

Indeed, the proof considers a backward iterate $P_{-s}(R)$ such that $d(P_{-s}(D_1), P_{-s}(D_2))$ is comparable to $\max(\text{Diam}(P_{-s}(D_1)), \text{Diam}(P_{-s}(D_2)))$. The second condition ensures that the backward iterates of R exist and remain small until such a time.

If we know that $R \subset B(0_y, \beta)$ for β small, this can be verified as follow: by the hyperbolicity for \mathcal{F} , the diameter of the \mathcal{F} -boundary of $P_{-s}(R)$ decreases exponentially as s increases; by the domination, the ratio between the diameter of the \mathcal{F} -boundary and the distance between the two \mathcal{F} -boundaries also increases exponentially. One deduces that the diameter of $P_{-s}(R)$ remains small until the first time s such that the \mathcal{F} -boundary of $P_{-s}(R)$ becomes much smaller than $d(P_{-s}(D_1), P_{-s}(D_2))$

4 Topological hyperbolicity

Standing assumptions. In the whole section, (\mathcal{N}, P) is a C^2 local fibered flow over a topological flow (K, φ) and π is an identification compatible with (P_t) on an open set U such that:

- (A1) there exists a dominated splitting $\mathcal{N} = \mathcal{E} \oplus \mathcal{F}$ and the fibers of \mathcal{E} are one-dimensional,
- (A2) \mathcal{E} is uniformly contracted on an open set V containing $K \setminus U$,
- (A3) \mathcal{E} is uniformly contracted over any periodic orbit $\mathcal{O} \subset K$.

From the last item and Proposition 3.17, the bundle \mathcal{E} is 2-dominated. By Theorem 3.19, one can fix a C^2 -plaque family \mathcal{W}^{cs} tangent to \mathcal{E} and by Theorem 3.20, there exists a C^1 -plaque family \mathcal{W}^{cu} for half generalized orbits parametrized by $(-\infty, 0]$ that are in a small neighborhood of K . Both are locally invariant by the time-one maps P_1 and \bar{P}_{-1} respectively.

The goal of this section is to prove the following theorem (see Subsection 4.7.4):

Theorem 4.1. *Under the assumptions above, one of the following properties occur:*

- *There exists a non empty proper invariant compact subset $K' \subset K$ such that $\mathcal{E}|_{K'}$ is not uniformly contracted.*
- *K is a normally expanded irrational torus.*
- *\mathcal{E} is topologically contracted: there is $\varepsilon_0 > 0$ such that the image $P_t(\mathcal{W}_{\varepsilon_0}^{cs}(x))$ is well-defined for any $t \geq 0$, $x \in K$, and*

$$\lim_{t \rightarrow +\infty} \sup_{x \in K} |(P_t(\mathcal{W}_{\varepsilon_0}^{cs}(x)))| = 0.$$

Choice of constants. Let us remark that by assumption (A2), the bundle \mathcal{E} is also uniformly contracted on a neighborhood of $\text{Closure}(V)$. Hence, on the set $\bigcup_{s \in [0, \varepsilon]} \varphi_s(V)$, for some $\varepsilon > 0$ small. By Remark 3.5.(a), one can rescale the time so that $\varepsilon = 1$ and assume:

- (A2') \mathcal{E} is uniformly contracted on $\bigcup_{s \in [0, 1]} \varphi_s(V)$, where V is an open set containing $K \setminus U$.

As introduced in Subsection 3.3.1 we denote by $\tau_0 \in \mathbb{N}$ and $\lambda > 1$ the constants associated to the 2-domination $\mathcal{E} \oplus \mathcal{F}$. Proposition 3.34 associates to the set $W := V$ the constants $T_{\mathcal{F}} \geq 0$ defining Pliss points for \mathcal{F} and $C_{\mathcal{E}}, \lambda_{\mathcal{E}}$ defining the hyperbolicity for \mathcal{E} . We also choose arbitrarily $\lambda_{\mathcal{F}} \in (1, \lambda)$. Sections 3.3 gives some constants $\alpha_{\mathcal{F}}$ controlling size of the unstable manifold at $(T_{\mathcal{F}}, \lambda_{\mathcal{F}})$ -Pliss points of K , and also at half generalized orbits parametrized by $(-\infty, 0]$ in the η -neighborhood of K , provided η is small enough. There exists $C_{\mathcal{F}} > 0$ such that any $(T_{\mathcal{F}}, \lambda_{\mathcal{F}})$ -Pliss generalized orbit \bar{u} is also $(C_{\mathcal{F}}, \lambda_{\mathcal{F}})$ -hyperbolic for \mathcal{F} . Proposition 3.38 gives $\alpha > 0$ such that for any generalized orbit in the η -neighborhood of K which is $(C_{\mathcal{F}}, \lambda_{\mathcal{F}})$ -hyperbolic for \mathcal{F} , the backward iterates $P_{-t}(\mathcal{W}_{\alpha}^{cu})$ have diameter smaller than $\alpha_{\mathcal{F}} \lambda_{\mathcal{F}}^{-t/2}$.

We also consider small constants $\beta_{\mathcal{F}}, r, \delta_0, \alpha' > 0$ which will be chosen in this order during this section: they control distances inside the spaces \mathcal{N}_x , K , or \mathcal{W}_x^{cs} .

4.1 Topological stability and δ -intervals

4.1.1 Dynamics of δ -intervals

We introduce a crucial notion for this section.

Definition 4.2. Consider $\delta \in (0, \delta_0]$. A curve $I \subset \mathcal{W}_x^{cs}$ (not reduced to a single point), for $x \in K$, is called a δ -interval if $0_x \in I$ and for any $t \geq 0$, one has

$$|P_{-t}(I)| \leq \delta.$$

One example of δ -interval is given by a periodic point $z \in K$ together with a non-trivial interval in $\mathcal{W}^{cs}(z)$ that is periodic for (P_t) and contains 0_z .

Definition 4.3. A δ -interval I is periodic if it coincides with $P_{-T}(I)$ for some $T > 0$. We say that a δ -interval I at x is contained in the unstable set of some periodic δ -interval if:

1. the α -limit set $\alpha(x) \subset K$ of x is the orbit of a periodic point y ,
2. y admits a periodic δ -interval \widehat{I}_y ,
3. $P_{-t}(I)$ accumulates as $t \rightarrow +\infty$ on the orbit of a (maybe trivial) interval $I_y \subset \widehat{I}_y$.

The next property will be proved in Section 4.6.

Lemma 4.4. There is $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0]$, for any periodic δ -interval $I \subset \mathcal{N}_q$, there exists $\chi > 0$ with the following property.

Let z be close to q , let $L \subset \mathcal{N}_z$ be an arc which is close to I in the Hausdorff topology and contains 0_z and let $T > 0$ such that $|P_{-t}(L)| \leq \delta$ for any $t \in [0, T]$. Then $|P_{-T}(L)| > \chi$.

Proposition 4.5 describes dynamics of δ -intervals. It is an analogue to [PS₂, Theorem 3.2].

Proposition 4.5. There is $\delta_0 > 0$ such that if there is a δ -interval $I \subset \mathcal{W}_x^{cs}$ for $\delta \in (0, \delta_0]$ then

- either K contains a normally expanded irrational torus,
- or I is contained in the unstable set of some periodic δ -interval.

Remark. In the first case one can even show that $\alpha(x)$ is a normally expanded irrational torus. We will not use it.

Strategy of the proof of Proposition 4.5. The next five subsections are devoted to the proof:

- One introduces a *limit* δ -interval I_∞ from the backward orbit of I (Section 4.2).
- I_∞ has returns close to itself (Section 4.3.3). Under some “non-shifting” condition one gets a periodic δ -interval (Section 4.3.2) and the last case of the proposition holds.
- If the “non-shifting” condition does not hold, there exists a normally expanded irrational torus which attracts x , I and I_∞ by backward iterations (Sections 4.4 and 4.5).

The conclusion of the proof is given in Section 4.6.

4.1.2 Topological stability

Before proving Lemma 4.4 and Proposition 4.5, we derive a consequence.

Proposition 4.6. If there is no normally expanded irrational torus, then \mathcal{E} is topologically stable: there is $\varepsilon_0 > 0$ and for any $\varepsilon_1 \in (0, \varepsilon_0)$, there is $\varepsilon_2 > 0$ such that

$$\forall x \in K \text{ and } \forall t > 0, \quad P_t(\mathcal{W}_{\varepsilon_2}^{cs}(x)) \subset \mathcal{W}_{\varepsilon_1}^{cs}(\varphi_t(x)).$$

Proof. By Remark 3.24 it is enough to check that $|P_t(\mathcal{W}_{\varepsilon_2}^{cs}(x))|$ is bounded by ε_1 . One can choose δ_0 small so that Proposition 4.5 holds. We argue by contradiction. If the topological stability does not hold, then there exist $\delta \in (0, \delta_0]$ a sequence (x_n) in K and, for each n , an interval $I_n \subset \mathcal{W}^{cs}(x_n)$ containing 0 and a time $T_n > 0$ such that:

- $|I_n| \rightarrow 0$ as $n \rightarrow +\infty$.
- $|P_{T_n}(I_n)| = \delta$ and $|P_t(I_n)| < \delta$ for all $0 < t < T_n$.

Taking a subsequence, one can assume that $(\varphi_{T_n}(x_n))$ converges to a point $x \in K$ and $(P_{T_n}(I_n))$ to an interval I . We have $|I| = \delta$ and $|P_{-t}(I)| \leq \delta$ for all $t > 0$, so that I is a δ -interval.

The second case of the proposition is satisfied by I and $L_n = P_{T_n}(I_n)$, n large. Lemma 4.4 implies that $P_{-T_n}(L_n)$ has length uniformly bounded away from 0. This contradicts the fact that the length of $I_n = P_{-T_n}(L_n)$ goes to 0 when $n \rightarrow \infty$. \square

4.2 Limit δ -interval I_∞

One can obtain infinitely many δ -intervals with length uniformly bounded away from zero at points of the backward orbit of a δ -interval. The goal of this section is to prove this property.

Proposition 4.7. *If $\delta > 0$ is small enough, for any $x \in K$ and any δ -interval I , there exists an increasing sequence (n_k) in \mathbb{N} and δ -intervals \widehat{I}_k at $\varphi_{-n_k}(x)$ such that:*

- $P_{-n_k}(I)$ and $P_{n_\ell - n_k}(\widehat{I}_\ell)$ are contained in \widehat{I}_k for any $\ell \leq k$,
- $\varphi_{-n_k}(x)$ is $T_{\mathcal{F}}$ -Pliss and belongs to $K \setminus V$, for any $k \geq 0$,
- (\widehat{I}_k) converges to some δ -interval I_∞ at some point $x_\infty \in K \setminus V$.

4.2.1 Existence of hyperbolic returns

Lemma 4.8. *If $\delta > 0$ is small enough, any point $x \in K$ which admits a δ -interval has infinitely many backward iterates $\varphi_{-n}(x)$, $n \in \mathbb{N}$, in $K \setminus V$ that are $T_{\mathcal{F}}$ -Pliss.*

Proof. Since the backward iterates $P_{-n}(I)$, $n \in \mathbb{N}$, of a δ -interval I are still δ -intervals, it is enough to show that any point x has at least one backward iterate by φ_1 in $K \setminus V$ that is $T_{\mathcal{F}}$ -Pliss. The proof is done by contradiction.

Let $\chi > 0$ such that $1 + \chi < \min(\lambda, \lambda_{\mathcal{E}})$. If $\delta_1 > 0$ is small enough, then for any δ -interval I with $\delta < \delta_1$, at any point x , we have

$$\|DP_{-t}|_{\mathcal{E}(x)}\| \leq (1 + \chi)^t \frac{|P_{-t}(I)|}{|I|} \leq \frac{(1 + \chi)^t \delta}{|I|}, \quad \forall t \geq 0.$$

With the domination estimate (7) of Section 3.3.1, one gets for any $k \geq 0$:

$$\prod_{j=0}^{k-1} \|DP_{-\tau_0}|_{\mathcal{F}(\varphi_{-j\tau_0}(x))}\| \leq \frac{(1 + \chi)^{k\tau_0} \lambda^{-2k\tau_0} \delta}{|I|}.$$

Using $(1 + \chi) < \lambda$ and Pliss lemma (see [M₂, Lemma 11.8]), there exists an arbitrarily large integer i such that for any $k \geq 0$ one has:

$$\prod_{j=0}^{k-1} \|DP_{-\tau_0}|_{\mathcal{F}(\varphi_{-(j+i)\tau_0}(x))}\| \leq \lambda^{-k\tau_0}.$$

This proves that x has arbitrarily large 0-Pliss backward iterates by φ_{τ_0} .

Let us fix any of these 0-Pliss backward iterates $\varphi_{-k\tau_0}(x)$. By contradiction, we assume that there is no iterate $\varphi_{-n}(x)$ in $K \setminus V$ that is $T_{\mathcal{F}}$ -Pliss with $1 \leq n \leq k\tau_0$. We can thus apply Proposition 3.34 with $W = V$; for any such n , one gets

$$\|DP_n|_{\mathcal{E}(\varphi_{-k\tau_0}(x))}\| \leq C_{\mathcal{E}} \lambda_{\mathcal{E}}^{-n}.$$

As before, this implies that

$$|I| \leq |P_{-k\tau_0}(I)| C_{\mathcal{E}} \lambda_{\mathcal{E}}^{-k\tau_0} (1 + \chi)^{k\tau_0} \leq \delta C_{\mathcal{E}} (1 + \chi)^{k\tau_0} \lambda_{\mathcal{E}}^{-k\tau_0}.$$

Since $(1 + \chi)/\lambda_{\mathcal{E}} < 1$ and k is arbitrarily large, one gets $|I| = 0$ which is a contradiction. \square

4.2.2 Rectangles associated to δ -intervals of Pliss points

When $\delta > 0$ is smaller than η , any point u in a δ interval I has a well defined backward orbit under (P_t) , which satisfies the definition of generalized orbit in the η -neighborhood of K . Consequently, it has a space $\mathcal{F}(u)$ and a center-unstable plaque $\mathcal{W}^{cu}(u)$. Moreover by Lemma 3.36, if the point $x \in K$ such that $u \in \mathcal{N}_x$ is $T_{\mathcal{F}}$ -Pliss, then u is $(T_{\mathcal{F}}, \lambda_{\mathcal{F}})$ -Pliss.

We will need to build a rectangle $R(I)$ foliated by unstable plaques for each δ -interval I above a Pliss point $x \in K$. As before d denotes the dimension of the fibers \mathcal{N}_x .

Proposition 4.9. *Fix $\beta_{\mathcal{F}} \in (0, \beta_0/4)$. There exist $\alpha_{min}, \delta_0, C_R > 0$ such that for any $\delta \in (0, \delta_0]$, any $T_{\mathcal{F}}$ -Pliss point $x \in K \setminus V$ and any δ -interval I at x , we associate a rectangle $R(I)$ which is the image of $[0, 1] \times B_{d-1}(0, 1)$ by a homeomorphism ψ such that:*

1. $\psi: [0, 1] \times \{0\} \rightarrow \mathcal{N}_x$ is a C^1 -parametrization of I ,
2. each $u \in R(I)$ belongs to a (unique) leaf $\psi(\{z\} \times B_{d-1}(0, 1))$ denoted by $W_{R(I)}^u(u)$; it is contained in $\mathcal{W}^{cu}(u)$ (and $\mathcal{W}^{cu}(\psi(z, 0))$), and it contains a disc with radius α_{min} ,
3. $\text{Volume}(R(I)) \geq C_R \cdot |I|$,
4. for any $t > 0$, $\text{Diam}(P_{-t}(W_{R(I)}^u(u))) < \beta_{\mathcal{F}} \lambda_{\mathcal{F}}^{-t/2}$; hence $P_{-t}(R(I)) \subset B(0, 2\beta_{\mathcal{F}}) \subset \mathcal{N}_{\varphi_{-t}(x)}$.

Moreover, if x, x' are $T_{\mathcal{F}}$ -Pliss points with δ -intervals I, I' and if $t, t' > 0$ satisfy:

- $\varphi_{-t}(x), \varphi_{-t'}(x')$ belong to $K \setminus V$ and are r_0 -close,
- $P_{-t}(R(I))$ and the projection of $P_{-t'}(R(I'))$ by $\pi_{\varphi_{-t}(x)}$ intersect,

then, the foliations of $P_{-t}(R(I))$ and $\pi_{\varphi_{-t}(x)} \circ P_{-t'}(R(I'))$ coincide: if $P_{-t}(W_{R(I)}^u(u))$ and $\pi_{\varphi_{-t}(x)} P_{-t'}(W_{R(I')}^u(u'))$ intersect, they are contained in a same C^1 -disc tangent to $\mathcal{C}^{\mathcal{F}}$.

Remark 4.10. Assume that $\beta_{\mathcal{F}}$ and $r > 0$ are small enough. By the Global invariance, under the assumptions of the last part of Proposition 4.9, and assuming $d(\varphi_{-t}(x), \varphi_{-t'}(x')) < r$, there exists $\theta \in \text{Lip}$ such that $\theta(0) = 0$ and

- the distance $d(\varphi_{-\theta(s)-t'}(x'), \varphi_{-s-t}(x))$ for $s > 0$ remains bounded (and arbitrarily small if $\beta_{\mathcal{F}} > 0$ and r have been chosen small enough),
- $P_{-s} \circ \pi_{\varphi_{-t}(x)} = \pi_{\varphi_{-\theta(s)-t}(x)} \circ P_{-\theta(s)}$ on $P_{-t'}(R(I'))$ when $\varphi_{-s-t}(x)$ and $\varphi_{-\theta(s)-t'}(x')$ are r_0 -close and belong to U .

Proof of Proposition 4.9. By Proposition 3.40, one associates to $C_{\mathcal{F}}, \lambda_{\mathcal{F}}$ some constants Δ, β . One can reduce $\beta_{\mathcal{F}}$ to ensure $2\beta_{\mathcal{F}} < \beta$.

By Proposition 3.38 one associates to $C_{\mathcal{F}}, \lambda_{\mathcal{F}}, T_{\mathcal{F}}$ and $\beta_{\mathcal{F}}$, some quantities α', η' . We will assume that δ is smaller than η' so that for any u in the δ -interval I and any $t \geq 0$ we have an exponential contraction of the unstable plaque of size α' :

$$\forall t \geq 0, \quad \text{Diam}(\bar{P}_{-t}(\mathcal{W}_{\alpha'}^{cu}(\bar{u}))) \leq \beta_{\mathcal{F}} \lambda_{\mathcal{F}}^{-t/2}.$$

One parametrizes the curve I by $[0, 1]$. From the plaque-family Theorem 3.20, there exists a continuous map $\psi: [0, 1] \times \mathbb{R}^{d-1} \rightarrow \mathcal{N}_x$ such that $\psi([0, 1] \times \{0\}) = I$. Up to rescale \mathbb{R}^{d-1} , the image $\psi(\{s\} \times B_{d-1}(0, 1))$ is small, hence is contained in $\mathcal{W}_{\alpha'}^{cu}(\psi(s, 0))$ for each $s \in [0, 1]$.

Note that two plaques $\psi(\{s\} \times B_{d-1}(0, 1)), \psi(\{s'\} \times B_{d-1}(0, 1))$, for $s \neq s'$, can not be contained in a same C^1 -disc tangent to $\mathcal{C}^{\mathcal{F}}$ since they intersect the transverse curve I at two different points. By coherence, they are disjoint: indeed since the backward orbit of x has arbitrarily large backward iterates in $K \setminus V$ (see Lemma 4.8), Proposition 3.25 applies.

Hence ψ is injective on $[0, 1] \times B_{d-1}(0, 1)$ and is a homeomorphism on its image $R(I)$ by the invariance of domain theorem. In particular $R(I)$ satisfies Definition 3.39 and is a rectangle. The coherence again implies that for any $u \in \psi(\{s\} \times B_{d-1}(0, 1))$, the plaque $\mathcal{W}_{\alpha'}^{cu}(u)$ contains $\psi(\{s\} \times B_{d-1}(0, 1))$. By compactness, there exists α_{\min} (which does not depend on x and I) such that $\psi(\{s\} \times B_{d-1}(0, 1))$ contains $\mathcal{W}_{\alpha_{\min}}^{cu}(\psi(s, 0))$ for any $s \in [0, 1]$.

The rectangle $R(I)$ has distortion bounded by Δ (from Proposition 3.40), hence one bounds $\text{Volume}(R(I))$ from below by using Fubini's theorem and integrating along curves tangent to $\mathcal{C}^{\mathcal{E}}$. This gives the four items of the lemma.

The last part is a direct consequence of the coherence (Proposition 3.25). \square

4.2.3 Maximal δ -intervals

In the setting of Proposition 4.7, let us consider all the integers $n_0 = 0 < n_1 < n_2 < \dots < n_k < \dots$ such that the backward iterate $x_k := \varphi_{-n_k}(x)$ belongs to $K \setminus V$ and is $T_{\mathcal{F}}$ -Pliss. We introduce inductively some maximal δ -intervals \hat{I}_k at these iterates such that $I \subset \hat{I}_0$ and $P_{n_k - n_{k+1}}(\hat{I}_k) \subset \hat{I}_{k+1}$. We denote $R_k = R(\hat{I}_k)$.

Lemma 3.30 associates to $C_{\mathcal{E}}, \lambda_{\mathcal{E}}$ some constants $C'_{\mathcal{E}}, \delta_E$. Let us assume $\delta < \delta_E$. From the definition of the sequence (n_k) , Proposition 3.34 implies that $(\varphi_{-n_{k+1}}(x), \varphi_{-n_k}(x))$ is $(C_{\mathcal{E}}, \lambda_{\mathcal{E}})$ -hyperbolic for \mathcal{E} and for each k . Lemma 3.30 then gives

$$\sum_{m=n_k}^{n_{k+1}} |P_{n_k - m}(\hat{I}_k)| \leq C'_{\mathcal{E}} |\hat{I}_{k+1}|. \quad (11)$$

4.2.4 Non-disjointness

Lemma 4.11 (Existence of intersections). *If $\delta_0 > 0$ is small enough, then for any $\delta \in (0, \delta_0]$, for any $r > 0$ there exist $k < \ell$ arbitrarily large such that $d(x_k, x_{\ell}) < r$ and the interior of $\pi_{x_k}(R_{\ell})$ and the interior of R_k intersect.*

Proof. Since $(\pi_{x,y})$ is a continuous family of diffeomorphisms on the open set U containing $K \setminus V$, the following property holds provided r is small enough:

For any $x, y \in K \setminus V$ with $d(x, y) < r$, the projection $\pi_y: \mathcal{N}_x \rightarrow \mathcal{N}_y$ satisfies:

$$\pi_y(B(0, \beta_0/2)) \subset B(0, \beta_0),$$

$$\det(D\pi_{x,y})(u) \leq 2 \text{ for any } u \in B(0, \beta_0/2).$$

By compactness, one can find a finite set $Z \subset U$ such that any $x \in K \setminus V$ satisfies $d(x, z) < r/2$ for some $z \in Z$. For each point x_k we associate some $z_k \in Z$ such that $d(x_k, z_k) < r/2$. Since $R_k \subset B(0, 2\beta_{\mathcal{F}}) \subset \mathcal{N}_{x_k}$ with $\beta_{\mathcal{F}} < \beta_0/4$ and from Proposition 4.9, we have

$$\pi_{z_k}(R_k) \subset B(0, \beta_0) \subset \mathcal{N}_{z_k},$$

$$\text{Volume}(\text{Interior}(\pi_{z_k}(R_k))) \geq \frac{1}{2} \text{Volume}(\text{Interior}(R_k)) \geq \frac{C_R}{2} |\widehat{I}_k|.$$

Let us assume by contradiction that the statement of the lemma does not hold. One deduces that there exists $s \geq 0$ such that for any $z \in Z$ and any $k, \ell \geq s$ such that $z_k = z_\ell = z$ we have

$$\pi_z(\text{Interior}(R_k)) \cap \pi_z(\text{Interior}(R_\ell)) = \emptyset.$$

In particular if C_{Vol} denotes the supremum of $\text{Volume}(B(0_x, \beta_0))$ over $x \in K$,

$$\sum_{k=1}^{\infty} |\widehat{I}_k| \leq 2C_R^{-1} C_{Vol} \text{Card}(Z).$$

With (11) we get for any k :

$$\sum_{m=0}^{+\infty} |P_{-m}(\widehat{I}_k)| \leq C'_{\mathcal{E}} \sum_{\ell=k}^{\infty} |\widehat{I}_\ell| \leq C_{Sum} := 2C_R^{-1} C_{Vol} \text{Card}(Z) C'_{\mathcal{E}}. \quad (12)$$

By Denjoy-Schwartz Lemma 3.27 one gets $\eta_S > 0$ and for k large one can introduce an interval $J \subset \mathcal{W}^{cs}(x_k)$ containing \widehat{I}_k and of length equal to $(1 + \eta_S)|\widehat{I}_k|$. One gets

$$|P_{-m}(J)| \leq 2|P_{-m}(\widehat{I}_k)|, \quad \forall m \geq 0.$$

From (12), $\sup_{m \geq 0} |P_{-m}(\widehat{I}_k)|$ is arbitrarily small for k large, hence $|P_{-t}(J)|$ is smaller than δ for any $t > 0$. This proves that J is a δ -interval, contradicting the maximality of \widehat{I}_k . \square

4.2.5 Non-shrinking property

Lemma 4.12. *If δ_0 is small enough, the length $|\widehat{I}_k|$ does not go to zero as $k \rightarrow \infty$.*

Proof. We first introduce an integer $N \geq 1$ such that $\beta_{\mathcal{F}} \lambda_{\mathcal{F}}^{N/2}$ is much smaller than α_{min} .

We argue by contradiction. Assume that $|\widehat{I}_k|$ is arbitrarily small as k is large. From (11) we deduce that for any $\delta' \in (0, \delta)$, if k is large enough, then \widehat{I}_k is a δ' -interval.

By Lemma 4.11, there exist $k \neq \ell$ large such that x_k, x_ℓ are arbitrarily close and we have that $\pi_{x_k}(\text{Interior}(R_\ell)) \cap \text{Interior}(R_k) \neq \emptyset$. By Remark 4.10, the unstable foliations of $\pi_{x_k}(R_\ell)$ and R_k coincide on the intersection, hence one of the following cases occurs (see Figure 1).

1. There exists an endpoint u of \widehat{I}_k such that $W_{R_k}^u(u)$ intersects $\pi_{x_k}(\widehat{I}_\ell)$ at a point which is not endpoint.
2. The endpoints of \widehat{I}_k and $\pi_{x_k}(\widehat{I}_\ell)$ have the same unstable manifolds.

In the first case, by Remark 4.10, there exists a homeomorphism θ of $[0, +\infty)$ such that $\varphi_{-t}(x_k)$ and $\varphi_{-\theta(t)}(x_\ell)$ are close for any $t > 0$, and $P_{-t}(\pi_{x_k}(R(I_\ell)))$ remains in a neighborhood of $\varphi_{-t}(x_k)$ which is arbitrarily small if δ' and $d(x_k, x_\ell)$ are small enough. The rectangle $\pi_{x_k}(R(\widehat{I}_\ell))$ intersects $\mathcal{W}^{cs}(x_k)$ along an interval J , which meets \widehat{I}_k . This proves that the union of J with \widehat{I}_k is a δ -interval and contradicts the maximality since J is not contained in \widehat{I}_k in this first case.

In the second case, without loss of generality, we assume $n_\ell > n_k$ and set $T := n_\ell - n_k$. We introduce the map $\tilde{P}_{-T} := \pi_{x_k} \circ P_{n_k - n_\ell}$. Since r has been chosen small enough and since the endpoints of \widehat{I}_k and $\pi_{x_k}(\widehat{I}_\ell)$ have the same unstable manifolds, the iterates $\tilde{P}_{-T}^i(\widehat{I}_k)$, $i \geq 0$, are all contained in R_k . Hence by the Global invariance, there exists a sequence of times $0 < t_1 < t_2 < \dots$ going to $+\infty$ such that

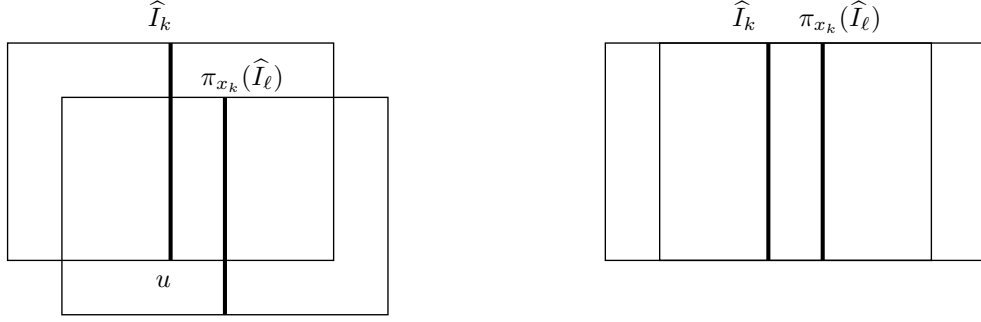


Figure 1

- $\varphi_{-t_1}(x_k) = x_\ell$,
- $(\varphi_{-t}(x_k))_{t_i \leq t \leq t_{i+1}}$ shadows $(\varphi_{-t}(x_k))_{0 \leq t \leq t_1}$,
- $\varphi_{-t_i}(x_k)$ is close to x_k and projects by π_{x_k} in R_k .

Note that the differences $t_{i+1} - t_i$ are uniformly bounded in i by some constant T_0 . Since δ' can be chosen arbitrarily small (provided k is large), for $t = t_i$ arbitrarily large, the interval $J = \pi_{\varphi_{-t_i}(x_k)}(R_k) \cap \mathcal{W}^{cs}(\varphi_{-t_i}(x_k))$ contains 0 and is a $\delta/2$ -interval. One can choose t_i and a backward iterate x_j such that $t_i \leq n_j \leq t_i + T_0$. Since $P_{t_i-n_j}(J)$ is a $\delta/2$ -interval and \hat{I}_j is a δ' -interval, $\hat{I}_j \cup P_{t_i-n_j}(J)$ is a δ -interval. As j can be chosen arbitrarily large, $|\hat{I}_j|$ is arbitrarily small, whereas $|P_{t_i-n_j}(J)|$ is uniformly bounded away from zero (since $n_j - t_i$ is bounded). Consequently $\hat{I}_j \cup P_{t_i-n_j}(J)$ is strictly larger than \hat{I}_j , contradicting the maximality. \square

4.2.6 Existence of limit intervals

The Proposition 4.7 now follows easily from the previous lemmas, up to extract a subsequence from the sequence of hyperbolic times (n_k) . \square

4.3 Returns of δ -intervals

4.3.1 Definition of returns and of shifting returns

We now introduce the times which will allow to induce the dynamics near a δ -interval.

Definition 4.13. Let $x \in K \setminus V$ be a $T_{\mathcal{F}}$ -Pliss point and I be a δ -interval at x . A time $t > 0$ is a return of I if

- $\varphi_{-t}(x)$ and x are r_0 -close, and $\text{Interior}(R(I)) \cap \text{Interior}(\pi_x \circ P_{-t}(R(I))) \neq \emptyset$,
- for any $z, z' \in I$ such that $\pi_x \circ P_{-t}(W_{R(I)}^u(z)) \cap W_{R(I)}^u(z') \neq \emptyset$, we have

$$\pi_x \circ P_{-t}(W_{R(I)}^u(z)) \subset W_{R(I)}^u(z').$$

We then denote by \tilde{P}_{-t} the map $\pi_x \circ P_{-t}: R(I) \rightarrow \mathcal{N}_x$.

A sequence of returns (t_n) is deep if $t_n \rightarrow +\infty$ and if one can find one sequence (x_n) in K with $\pi_x(x_n) \in R(I)$ such that $\tilde{P}_{-t_n} \circ \pi_x(x_n) \rightarrow 0_x$ as $n \rightarrow +\infty$.

Remark. When (t_n) is a sequence of deep returns, $\tilde{P}_{-t_n} \circ \pi_x(R(I))$ gets arbitrarily close to $\mathcal{W}^{cs}(x)$ as $n \rightarrow +\infty$. Indeed, since t is large, $\tilde{P}_{-t_n} \circ \pi_x(R(I))$ is thin and contained in a small neighborhood of $\pi_x(\mathcal{W}^{cs}(\varphi_{-t'_n}(x_n)))$, where $\tilde{P}_{-t_n} \circ \pi_x(x_n) = \pi_x(\varphi_{-t'_n}(x_n))$. Moreover as $\tilde{P}_{-t_n} \circ \pi_x(x_n) \rightarrow 0_x$, the plaque $\pi_x(\mathcal{W}^{cs}(\varphi_{-t'_n}(x_n)))$ gets close to $\mathcal{W}^{cs}(x)$.

Lemma 4.14. *If δ_0 is small enough, a $T_{\mathcal{F}}$ -Pliss point $x \in K \setminus V$ with a δ -interval I for $\delta \in (0, \delta_0]$, some $t > 2 \log(\beta_{\mathcal{F}}/\delta_0)/\log(\lambda_{\mathcal{F}})$ satisfying $d(x, \varphi_{-t}(x)) < r_0$ and $z \in I$ satisfying $\pi_x \circ P_{-t}(z) \in \text{Interior}(R(I))$ and $d(\pi_x \circ P_{-t}(z), I) < \delta_0$, then the time t is a return.*

Proof. The leaves $W_{R(I)}^u(z)$ are tangent to $\mathcal{C}^{\mathcal{F}}$ and have uniform size α_{\min} . The interval I is tangent to $\mathcal{C}^{\mathcal{E}}$ and has size less than δ_0 , chosen much smaller than α_{\min} . Assuming that $t > 0$ is large enough, the images $P_{-t}(W_{R(I)}^u(z))$ have diameter smaller than $\beta_{\mathcal{F}}\lambda_{\mathcal{F}}^{t/2} < \delta_0$, so $P_{-t}(R(I))$ has diameter smaller than $2\delta_0$. If $d(\pi_x \circ P_{-t}(z), I)$ is smaller than δ_0 , the images $P_{-t}(W_{R(I)}^u(z))$ can not intersect the boundary of the leaves $W_{R(I)}^u(z')$. From the last part of Proposition 4.9, we get that $P_{-t}(W_{R(I)}^u(z))$ is disjoint or contained in $W_{R(I)}^u(z')$ for each $z, z' \in I$. \square

To each return, one associates a one-dimensional map $S_{-t} : I \rightarrow \mathcal{W}_x^{\text{cs}}$ as follows.

Proposition 4.15. *Assume that $\beta_{\mathcal{F}}, r, \delta_0$ are small enough and that δ is much smaller than δ_0 . Consider a return t of a δ -interval $I \subset \mathcal{N}_x$ and $d(\varphi_{-t}(x), x) < r$.*

Then there exists a δ_0 -interval $J \subset \mathcal{W}_x^{\text{cs}}$ containing I and a continuous injective map $S_{-t} : I \rightarrow J$ such that for each $u \in I$, the point $\tilde{P}_{-t}(u)$ belongs to $W_{R(J)}^u(S_{-t}(u))$.

Notice that from Definition 4.13, $S_{-t}(I)$ intersects I along a non-trivial interval.

Proof. Assuming $\beta_{\mathcal{F}}, r$ small enough, By Remark 4.10, the backward orbits of $\varphi_{-t}(x)$ and x stay at an arbitrarily small distance. The δ -interval $P_{-t}(I)$ projects by π_x on a set X whose backward iterates by P_{-s} are contained in $B(0_{\varphi_{-s}(x)}, \delta_0/2)$ (using the Global invariance). Since x is a $T_{\mathcal{F}}$ -Pliss point, any $u \in X$ is $(T_{\mathcal{F}}, \lambda_{\mathcal{F}})$ -Pliss (see Lemma 3.36) and has an unstable plaque $\mathcal{W}_{\alpha}^{cu}(u)$ whose backward iterates under P_{-t} have diameter smaller than $\alpha_{\mathcal{F}}\lambda_{\mathcal{F}}^{-t/2}$. One can thus project the arc X along the plaques $\mathcal{W}_{\alpha}^{cu}(u)$ of points $u \in X$ to a connected set $I' \subset \mathcal{W}_x^{\text{cs}}$ which intersects I . If δ_0 is small enough, and $P_{-s}(I \cup I')$ has diameter smaller than $2 \text{Diam}(P_{-s}(R(I) \cup \tilde{P}_{-t}(I))) < \delta_0$. Thus $J = I \cup I'$ is a δ_0 -interval.

By the coherence (Proposition 4.9, item 2), the plaques $\mathcal{W}_{\alpha}^{cu}(u)$ of $u \in X$ intersect $R(J)$ along a leaf $W_{R(J)}^u(u)$. Note that each plaque intersect $I' \subset J$ by construction. Moreover since $d(x, \varphi_{-t}(x))$ is small, X does not intersect the boundary of the leaves $W_{R(J)}^u(u)$. Thus $X \subset R(J)$.

Any point in $X = \tilde{P}_{-t}(I)$ can thus be projected to J along the leaves of $R(J)$. The map S_{-t} is the composition of this projection with \tilde{P}_{-t} . \square

Deepness has been introduced for the following statement.

Lemma 4.16. *Let (t_n) be a deep sequence of returns of a δ -interval I and let J be a δ_0 -interval containing I such that $S_{-t_n}(I) \subset J$ for each n . Then, there exists $n_0 \geq 1$ with the following property. If $n(1), \dots, n(\ell)$ is a sequence of integers with $n(i) \geq n_0$ and if there exists u in the interior of J satisfying for each $1 \leq i \leq \ell$*

$$S_{-t_{n(i)}} \circ \dots \circ S_{-t_{n(0)}}(u) \in \text{Interior}(J),$$

then there exists a return $t > 0$ such that $S_{-t} = S_{-t_{n(\ell)}} \circ \dots \circ S_{-t_{n(0)}}$.

The return t will be called *composition* of the returns $t_{n(0)}, \dots, t_{n(\ell)}$.

Proof. Note first that the image $\tilde{P}_{-t}(R(I))$ associated to a large return t has diameter smaller than 2δ . So if this set contains a point $\tilde{P}_{-t}(u)$ δ' -close to I , then $\tilde{P}_{-t}(R(I))$ is contained in the $2\delta + \delta'$ -neighborhood of I .

The lemma is proved by induction. The composition $S_{-t_{n(\ell-1)}} \circ \dots \circ S_{-t_{n(0)}}$ is associated to a return $t' > 0$. The point $\tilde{P}_{-t'}(u)$ belongs to the image $\tilde{P}_{-t_{n(\ell-1)}}(R(I))$, hence (since $t_{n(\ell-1)}$ is

large by deepness and Remark 4.3.1), is 2δ -close to I . By the Local injectivity and Remark 4.10, there exists an increasing homeomorphism θ such that $|\theta(0)| \leq 1/4$ and $\varphi_{\theta(-s)-t'}(x)$ shadows $\varphi_{-s}(x)$. Hence for $t = t' + \theta(t_n(\ell))$ the point $\varphi_{-t}(x)$ is close to x and $\tilde{P}_{-t} = \tilde{P}_{-t_n(\ell)} \circ \tilde{P}_{-t'}$ by the Global invariance. The first item of Definition 4.13 is satisfied.

The second one is implied by Proposition 4.9: if $\tilde{P}_{-t}(W_{R(I)}^u(z))$ intersects $W_{R(I)}^u(z')$, then these two discs match. The set $\tilde{P}_{-t}(R(I))$ has diameter smaller than 2δ ; it contains the point $\tilde{P}_{-t_n(\ell)} \circ \tilde{P}_{-t'}(u)$; this last point also belongs to $\tilde{P}_{-t_n(\ell)}(R(I))$ which is included in the 2δ -neighborhood of I . Hence $\tilde{P}_{-t}(R(I))$ is contained in the 4δ -neighborhood of I and can not intersect the boundary of the disc $W_{R(I)}^u(z')$. This gives $\tilde{P}_{-t}(W_{R(I)}^u(z)) \subset W_{R(I)}^u(z')$.

This proves that t is a return such that $\tilde{P}_{-t} = \tilde{P}_{-t_n(\ell)} \circ \tilde{P}_{-t'}$: the one-dimensional map associated to t coincides with the composition of the one-dimensional maps of the returns $t_n(\ell)$ and t' as required. \square

Definition 4.17. A return t is *shifting* if the one-dimensional map S_{-t} has no fixed point.

Let us fix an orientation on $\mathcal{W}^{cs}(x)$. It is preserved by S_{-t} when t is shifting.

A return shifts to the right (resp. to the left) if it is a shifting return and if there exists $u \in I$ that can be joined to $S_{-t}(u)$ by a positive arc (resp. negative arc) of $\mathcal{W}^{cs}(x)$.

4.3.2 Criterion for the existence of periodic δ -intervals

The following proposition shows that under the setting of Proposition 4.7, if the interval I has a large non-shifting return, then I is contained in the unstable set of some periodic δ -interval.

Proposition 4.18. Let I be a δ -interval at a point $x \in K$ and let J be a 3δ -interval at a $T_{\mathcal{F}}$ -Pliss point $y \in K \setminus V$ having large non-shifting returns. If $\pi_y \circ P_{-s}(I)$ intersects $R(J)$ for some $s \geq 0$, then I is contained in the unstable set of some periodic δ -interval.

Proof. By assumption there exist $t > 0$ large and $u \in J$ such that $W_{R(J)}^u(u)$ is mapped into itself by \tilde{P}_{-t} . This implies that \tilde{P}_{-t} has a fixed point v in $R(J)$.

Since t is a large return, assuming the $\beta_{\mathcal{F}}$ is small enough, there exists (by the local injectivity) $t_1 \in [t - 1/4, t + 1/4]$ such that $d(\varphi_{-t_1}(y), y) < r/2$. This allows (up to modify t) to assume that $d(\varphi_{-t}(y), y) < r/2$.

Claim. There is a periodic point $q \in K$ that is r_0 close to y such that $\pi_y(q)$ is a fixed point of \tilde{P}_{-t} in $R(J)$.

Proof of the claim. We build inductively a homeomorphism θ of $[0, +\infty)$ such that:

- for each $k \geq 0$ and each $s \in [0, t]$, we have $d(\varphi_{-\theta(s)}(y), \varphi_{-\theta(kt+s)}(y)) < r_0/2$,
- $d(\varphi_{-t_k}(y), y) < r$ where $t_k := \theta(kt)$ and r is the constant in Remark 4.10,
- we have $\pi_y \circ P_{-t_k} = (\tilde{P}_{-t})^k$.

Since $d(\varphi_{-t}(y), y) < r/2$, one defines $t_1 = t$ and $\theta(s) = s$ for $s \in [0, t]$.

Let us then assume that θ has been built on $[0, kt]$. Since $P_{-s}(v) \in P_{-s}(R(J))$ is $\beta_{\mathcal{F}}$ -close to $0_{\varphi_{-s}(y)}$ for any $s \geq 0$ and since $\pi_{\varphi_{-t_k}(y)}(v) = P_{-t_k}(v)$, Remark 4.10 applies and defines the homeomorphism θ on $[kt, (k+1)t]$ such that $d(\varphi_{-s}(y), \varphi_{-\theta(kt+s)}(y)) < r_0/4$ for $s \in [0, t]$. By the local injectivity, one can choose t_{k+1} with $|t_{k+1} - \theta(kt)| \leq 1/4$, $d(y, \varphi_{t_{k+1}}(y)) < r$ and one can modify θ near $(k+1)t$ so that $\theta((k+1)t) = t_{k+1}$.

Since $t_1 = t$ is large, and since $d(\varphi_{-\theta(kt+s)}(y), \varphi_{-\theta((k+1)t+s)}(y)) < r_0$, the No shear property (Proposition 3.6) implies inductively $t_k - t_{k-1} \geq 2$ for each $k \geq 1$. In particular $t_k \rightarrow +\infty$.

By the dominated splitting, the limit set Λ of the curves $\pi_y \circ P_{-t_k}(J) = \tilde{P}_{-t}^k(J)$ is a union of (uniformly Lipschitz) curves in \mathcal{N}_y , containing v and tangent to $\mathcal{C}^\mathcal{E}(y)$.

One can apply Proposition 3.23 to a periodic sequence of diffeomorphisms $f_0, \dots, f_{[t]}$ where f_i coincides with P_{-1} on $\mathcal{N}_{\varphi_{-m}(y)}$ for $0 \leq i < t-1$ and $f_{[t]}$ coincides with $\pi_y \circ P_{t-[t]+1}$ on $\mathcal{N}_{\varphi_{-[t]+1}(y)}$. We have $\tilde{P}_{-t} = f_{[t]} \circ \dots \circ f_0$ and any curve in the limit set Λ remain bounded and tangent to $\mathcal{C}^\mathcal{E}(y)$ under iterations by $(\tilde{P}_{-t})^{-1}$. Consequently, they are all contained in a same Lipschitz arc, invariant by \tilde{P}_{-t} . One deduces that a subsequence of $(\tilde{P}_{-t})^k(0_y)$ converges in \mathcal{N}_y to a fixed point p of $R(J)$.

By Lemma 3.7, there exists a periodic point $q \in K$ such that $\pi_y(q) = p \in R(J)$ is the fixed point of \tilde{P}_{-t} . \square

Now we prove that I is contained in the unstable set of some periodic δ -interval. Without loss of generality, we take $s = 0$. Since $\pi_y(q)$ and $\pi_y(x)$ belong to $R(J)$, by choosing $\beta_{\mathcal{F}}$ small enough and by the local invariance, one can assume that the distances $d(y, q)$ and $d(y, x)$ are smaller than any given constant. In particular, the projection by π_y of center-unstable plaques $\mathcal{W}^{cu}(u)$ of point $u \in \mathcal{N}_q$ is tangent to the cone $\mathcal{C}^\mathcal{F}$.

Claim. *There exists a periodic point $q_0 \in K$ such that $\alpha(x)$ is the orbit of q_0 .*

Proof of the Claim. By the assumptions, there is $u_0 \in I$ such that $\pi_y(u_0)$ is contained in $R(J)$. Since $\beta_{\mathcal{F}}$ and δ are small, by the Global invariance, there is $\theta_y \in \text{Lip}$ such that $|\theta_y(0)| \leq 1/4$ and $d(\varphi_{-s}(x), \varphi_{-\theta_y(s)}(y)) < r_0/2$ for any $s > 0$.

In a similar way, there exists $\theta_q \in \text{Lip}$ such that $|\theta_q(0)| \leq 1/4$ and $d(\varphi_{-t}(y), \varphi_{-\theta_q(t)}(q))$ remains small for any $t > 0$. By defining $\theta = \theta_q \circ \theta_y$, one deduces that $d(\varphi_{-\theta(t)}(q), \varphi_{-t}(x))$ is small too for any $t > 0$.

By using the Global invariance twice, $\|P_{-t}(\pi_y(x))\|$ remains small in the fiber $\mathcal{N}_{\varphi_{-t}(y)}$ and $\|P_{-t}(\pi_q(x))\|$ remains small in the fiber $\mathcal{N}_{\varphi_{-t}(q)}$. In particular $(P_{-t}(\pi_q(x)))_{t>0}$ is a half generalized orbit and has a center unstable plaque $\mathcal{W}^{cu}(\pi_q(x))$.

Let us first assume that $0_q \in \mathcal{W}^{cu}(\pi_q(x))$. Since $\|P_{-t}(\pi_q(x))\|$ remains small when $t \rightarrow +\infty$, the local invariance of the plaque families implies that $0_{\varphi_{-t}(q)} \in P_{-t}(\mathcal{W}^{cu}(\pi_q(x)))$ for any $t > 0$. Projecting to the fiber of $\varphi_{-\theta_q^{-1}(t)}(y)$, and using the Global invariance, one deduces that $P_{-t}(\pi_y(x))$ and $P_{-t}(\pi_y(q))$ is connected by a small arc tangent to $\mathcal{C}^\mathcal{F}$ for any $t > 0$. By Proposition 3.23, this shows that $\pi_y(x)$ and $\pi_y(q)$ belong to a same leaf $W_{R(J)}^u(u)$ of $R(J)$. Consequently, $d(P_{-t}(\pi_y(x)), P_{-t}(\pi_y(q))) \rightarrow 0$ as $t \rightarrow +\infty$. The Global invariance shows that $d(P_{-t}(\pi_q(x)), 0) \rightarrow 0$ as $t \rightarrow +\infty$. The Local injectivity then implies that $\varphi_{-t}(x)$ converges to the orbit of q .

If $\mathcal{W}^{cu}(\pi_q(x))$ does not contains 0_q , by Proposition 3.26, there exists a point $p \in \mathcal{N}_q$ which is fixed by P_{-2T} (where T is the period of q), such that $P_{-t}(\pi_q(x))$ converges to the orbit of p as $t \rightarrow +\infty$. By the Global invariance, $P_{-\ell T}(\pi_q(x))$ coincides with $\pi_q(\varphi_{-\theta^{-1}(\ell T)}(x))$ for $\ell \in \mathbb{N}$. Lemma 3.7 implies that there exists a periodic point $q_0 \in K$ such that $\varphi_{-\theta^{-1}(\ell T)}(x)$ converges to q_0 as $\ell \rightarrow +\infty$. Since θ is a bi-Lipschitz homeomorphism, one deduces that the backward orbit of x converges to the orbit of the periodic point q_0 . \square

Up to replace x and I by large backward iterates, there is $\theta \in \text{Lip}$ s.t. $d(\varphi_{-t}(x), \varphi_{-\theta(t)}(q_0))$ is small for any $t > 0$ and by the Global invariance, the intervals $P_{-t} \circ \pi_{q_0}(I)$ are curves tangent to the cone $\mathcal{C}^\mathcal{E}(\varphi_{-t}(q_0))$ which remain small for any $t > 0$. When $t = \ell T'$ where T' is the period of q_0 , they converge to a limit set which is a union of curves tangent to the cone $\mathcal{C}^\mathcal{E}(q_0)$ and contain 0_{q_0} . By Proposition 3.23, this limit set is an interval $I_{q_0} \subset \mathcal{W}^{cs}(q_0)$ that is fixed by $P_{T'}$. By the Global invariance, I_{q_0} is the limit of $\pi_y(P_{-\theta^{-1}(\ell T')}(I))$ as $\ell \rightarrow +\infty$. This implies that the backward orbit of I converges to the orbit of the δ -interval I_{q_0} . \square

4.3.3 Returns of limit intervals

We now prove that the interval I_∞ has always returns. Moreover, in the case I_∞ is not contained in a 3δ -interval having arbitrarily large non-shifting return (so that Proposition 4.18 can not be applied to some \widehat{I}_k and an interval J containing I_∞), we also prove that there exist shifting returns for I_∞ , both to the right and to the left.

Proposition 4.19. *Under the setting of Proposition 4.7,*

- either I_∞ is contained in a 3δ -interval having arbitrarily large non-shifting return,
- or I_∞ has two deep sequences of shifting returns one shifting to the left and the other one to the right.

Proof. We assume that the first case does not hold and one chooses an orientation, hence an order on $\mathcal{W}_{x_\infty}^{cs}$. Denote by $a_\infty < b_\infty$ the endpoints of I_∞ . There exists a $\frac{3}{2}\delta$ -interval L in $\mathcal{W}_{x_\infty}^{cs}$ such that $R(L)$ contains all the $\pi_{x_\infty}(\widehat{I}_k)$, k large. One can thus project $\pi_{x_\infty}(\widehat{I}_k)$ to $\mathcal{W}_{x_\infty}^{cs}$ by the unstable holonomy and denotes $a_k < b_k$ the endpoints of this projection. In the same way one denotes $c_k \in [a_k, b_k]$ the projection of $\pi_{x_\infty}(x_k)$. By Lemma 4.14, for any $k < \ell$ such that k and $\ell - k$ are large, there exists a (large) return $t > 0$ such that $\tilde{P}_{-t}(\pi_{x_\infty}(x_k)) = \pi_{x_\infty}(x_\ell)$. Since all large returns of L are shifting, such iterates x_k, x_ℓ of x do not project by π_{x_∞} to a same unstable manifold. Hence one can assume without loss of generality that for each $k < \ell$ we have $c_k > c_\ell$.

Claim. I_∞ admits a deep sequence of returns $t > 0$ shifting to the left.

Proof. For each $k < \ell$, there exists a return $t > 0$ of L such that $\tilde{P}_{-t}(\pi_{x_\infty}(x_k)) = \pi_{x_\infty}(x_\ell)$. This return shifts to the left. We claim that it is a return for I_∞ . Denote $a'_k = S_{-t}(a_k)$. The return t has been chosen so that $\pi_{x_\infty} \circ P_{-n_\ell+n_k}(\widehat{I}_k) = \tilde{P}_{-t} \circ \pi_{x_\infty}(\widehat{I}_k)$. Since \widehat{I}_ℓ contains $P_{-n_\ell+n_k}(\widehat{I}_k)$, this gives $a_\ell \leq a'_k$ and in particular $a_\ell < a_k$. Repeating this argument, one gets a decreasing subsequence (a_i) containing a_k and a_ℓ . It converges to a_∞ so that $a_\infty < a'_k < a_k$. This implies that both a_k and a'_k belong to I_∞ and that $t > 0$ is also a return for I_∞ . Note that when k is large and ℓ much larger, the time $t > 0$ is large and the intervals $\pi_{x_\infty}(\widehat{I}_k), \pi_{x_\infty}(\widehat{I}_\ell)$ are close to $\mathcal{W}_{x_\infty}^{cs}$. In particular the sequence of returns $t > 0$ one obtains by this construction is deep. \square

Let us fix a return t shifting left as given by the previous claim. We then choose $k \geq 1$ large and ℓ much larger and build a return which shifts to the right. As explained above we have $a_\infty < a_\ell < a_k$. Let us denote by $\bar{a}_\infty < \bar{a}_\ell < \bar{a}_k$ their images by S_{-t} . Since S_{-t} shifts left, for k large \bar{a}_k which is close to \bar{a}_∞ satisfies $\bar{a}_k < a_\infty$. See Figure 2. Let us denote $t' > 0$ a time such that the pieces of orbits $\varphi_{[-t',0]}(x_k)$ and $\varphi_{[-t,0]}(x_\infty)$ remain close (up to reparametrization), so that $\pi_{x_\infty} \circ \varphi_{-t'}(x_k) = \tilde{P}_{-t} \circ \pi_{x_\infty}(x_k)$.

The rectangle associated to the 3δ -interval $L \cup S_{-t}(L)$ contains $\pi_{x_\infty} \circ \varphi_{-t'}(x_k)$ and $\pi_{x_\infty}(x_\ell)$. In particular there exists a return $s > 0$ such that $\tilde{P}_{-s}(\pi_{x_\infty} \circ \varphi_{-t'}(x_k)) = \pi_{x_\infty}(x_\ell)$. Since $\ell - k$ is large, s is large and is a shifting return by our assumptions. We denote $a'_\infty < a'_k$ the images of $\bar{a}_\infty < \bar{a}_k$ by S_{-s} . Note that there exists a time $s' > 0$ such that $\pi_{x_\infty} \circ \varphi_{-s'}(\varphi_{-t'}(x_k)) = \tilde{P}_{-s} \circ \pi_{x_\infty}(P_{-t'}(x_k)) = \pi_{x_\infty}(x_\ell)$. By the local injectivity, one can choose s' such that $t' + s' = n_\ell - n_k$. In particular $S_{-s} \circ S_{-t}$ coincides with the one-dimensional map $S_{-\tilde{t}}$ associated to the return $\tilde{t} > 0$ sending $\pi_{x_\infty}(x_k)$ to $\pi_{x_\infty}(x_\ell)$. Note that t' is a return as considered in the proof of the previous claim; in particular we have proved that $a_\infty < S_{-\tilde{t}}(a_k) < a_k$. Hence $a_\infty < a'_k < a_k$.

We have obtained $\bar{a}_k < a_\infty < a'_k$, so that S_{-s} shifts to the right and $a_\infty < a'_\infty$. On the other hand $a'_\infty < a'_k < a_k < b_\infty$. So this gives $a'_\infty \in (a_\infty, b_\infty)$ which implies that S_{-s} is a return of I_∞ which shifts to the right as required. Since $\pi_{x_\infty} \circ \varphi_{-t'}(x_k) \in R(I)$ and $\pi_{x_\infty}(x_\ell) \rightarrow 0_{x_\infty}$, the sequence of returns s one may build by this construction is deep. \square

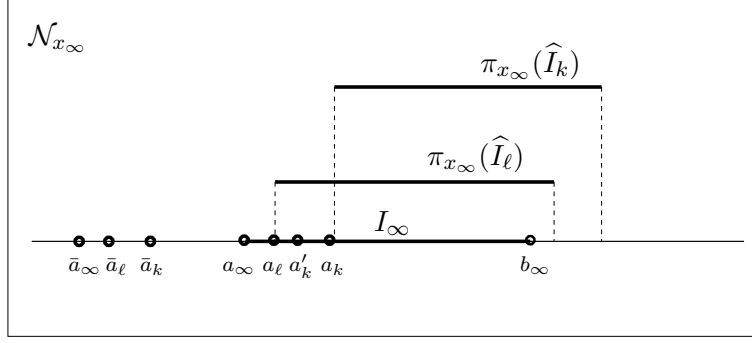


Figure 2

4.4 Aperiodic δ -intervals

We introduce the δ -intervals which will produce normally expanded irrational tori.

Definition 4.20. A δ -interval J at $x \in K \setminus V$ is aperiodic if there exist returns $t_1, t_2 > 0$ and intervals $J_1, J_2 \subset J$ such that:

- J_1, J_2 have disjoint interior and $J = J_1 \cup J_2$,
- $\tilde{P}_{-t_1}(J_1), \tilde{P}_{-t_2}(J_2)$ have disjoint interior and $J = \tilde{P}_{-t_1}(J_1) \cup \tilde{P}_{-t_2}(J_2)$,
- any non-empty compact set $\Lambda \subset J$ such that $\tilde{P}_{-t_1}(\Lambda \cap J_1) \subset \Lambda$ and $\tilde{P}_{-t_2}(\Lambda \cap J_2) \subset \Lambda$ coincides with J .

We prove here that the second case of the Proposition 4.7 gives aperiodic δ -intervals.

Proposition 4.21. Let $x \in K \setminus V$ be a $T_{\mathcal{F}}$ -Pliss point and let I be a δ -interval whose large returns are all shifting and which admit a deep sequence of returns (t_n^l) shifting to the left and another one (t_n^r) shifting to the right.

Then $\alpha(x)$ contains a point $x' \in K \setminus V$ having an aperiodic δ -interval.

We first select good returns for I .

Lemma 4.22. Under the setting of Proposition 4.21 there exists a δ -interval $L \subset I$, returns $t_1, t_2 > 0$ and intervals $L_1, L_2 \subset L$ such that

- L_1, L_2 have disjoint interiors and $L = L_1 \cup L_2$,
- $S_{-t_1}(L_1), S_{-t_2}(L_2)$ have disjoint interior and $L = S_{-t_1}(L_1) \cup S_{-t_2}(L_2)$,
- any non-empty compact set $\Lambda \subset L$ such that $S_{-t_1}(\Lambda \cap L_1) \subset \Lambda$ and $S_{-t_2}(\Lambda \cap L_2) \subset \Lambda$ coincides with L .

Moreover t_1, t_2 are composition of large returns inside the sequences (t_n^l) and (t_n^r) .

Proof. Let us consider two large returns $s, t > 0$ shifting to the right and to the left respectively. They can be chosen inside the sequences (t_n^r) and (t_n^l) , hence by Lemma 4.16 the compositions of the maps S_{-s}, S_{-t} , when they are defined, have no fixed point in I . Let us denote $D_s = I \cap S_{-s}^{-1}(I)$ and $I_s = S_{-s}(D_s)$ and similarly let us denote D_t, I_t the domain and the image of S_{-t} in I .

Step 1. One will reduce the interval I so that the assumptions of the Proposition 4.21 still hold but moreover either $D_s \cup D_t$ or $I_s \cup I_t$ coincides with I . See Figure 3.

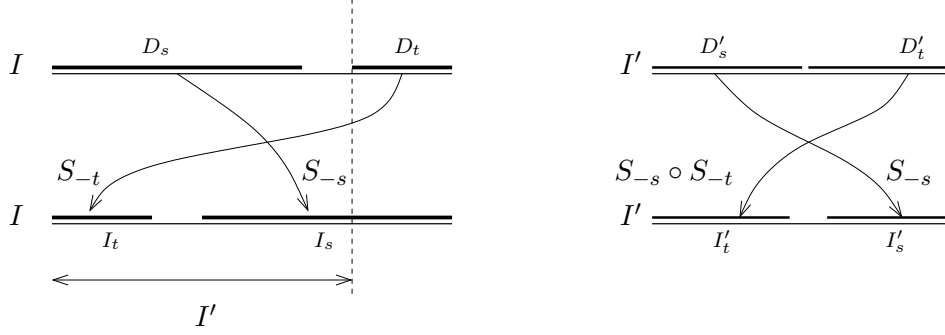


Figure 3

Note that both of these two sets contain the endpoints of I . If both are not connected one reduces I in the following way. Without loss of generality 0_x does not belong to D_t . One can thus move the right point of I (and D_t) inside D_t to the left and reduce I which still contains 0_x . This implies that the right endpoints of D_s, D_t, I_s, I_t, I move to the left whereas the left endpoints remain unchanged. At some moment one of the two intervals D_s, D_t becomes trivial. We obtain this way new intervals $I', D'_s, D'_t, I'_s, I'_t$. Note that both can not be trivial simultaneously since otherwise $S_{-s} \circ S_{-t}$ preserves the right endpoint of the new interval: this one dimensional map is associated to a large return of I which has a fixed point - a contradiction.

Let us assume for instance that D'_t (and I'_t) has become a trivial interval (the case D'_s is trivial is similar): the interval I' is bounded by the left endpoint of I and the left endpoint of D_t . Moreover the map S_{-t} sends the right endpoint of I' to its left endpoint. The map $S_{-t} \circ S_{-s}$ is associated to a return t' of I' which sends the right endpoint of D'_s to the left endpoint of I' , hence which shifts left and whose domain coincides with $I' \setminus D'_s$. We keep the return s . We have shown that $D'_s \cup D'_{t'}$ coincides with I' .

Step 2. One again reduces I so that the assumptions of Proposition 4.21 still hold, still $D_s \cup D_t$ or $I_s \cup I_t$ coincide with I and furthermore D_s, D_t (resp. I_s, I_t) have disjoint interior.

One follows the same argument as in step 1. For instance one moves the right endpoint of I to the left. Different cases can occur:

- the point x becomes the right endpoint of I , we then exchange the orientation of I and reduce I again; this case will not appear anymore;
- the new domains or the new images have disjoint interior; when this occurs either $D_s \cup D_t$ or $I_s \cup I_t$ coincide with I .

Note that the domains D_s, D_t can not become trivial as in step 1. Indeed if for instance D_t (and I_t) becomes trivial, since I still coincides with $D_s \cup D_t$ or $I_s \cup I_t$, one gets that I coincides with D_s or I_s , proving that $R(I)$ contains a fixed unstable manifold - a contradiction.

Step 3. We have now obtained a δ -interval $L \subset I$ and two returns t_1, t_2 , shifting to the left and the right respectively, whose domains L_1, L_2 and images $S_1(L_1), S_2(L_2)$ by $S_1 := S_{-t_1}$ and $S_2 := S_{-t_2}$ have disjoint interior and which satisfy one of the two cases:

Case 1. $L_1 \cup L_2 = L$,

Case 2. $S_1(L_1) \cup S_2(L_2) = L$.

After identifying the two endpoints of L , one can define an increasing map f on the circle which coincides with S_1 (resp. S_2) on the interior of I_1 (resp. I_2): in the first case it is injective and has one discontinuity (and f^{-1} is continuous) whereas in the second case f is continuous. By

our assumptions on s and t , f has no periodic point. We will prove that f is a homeomorphism conjugated to a minimal rotation. This will conclude the proof of the lemma.

We discuss the case 2 (the first case is very similar, arguing with f^{-1} instead of f). Poincaré theory of orientation preserving circle homeomorphisms extends to continuous increasing maps. Since f has no periodic point, there exists a unique minimal set K . Let us assume by contradiction that K is not the whole circle. Let J be a component of its complement. It is disjoint from its preimages $J_{-n} = f^{-n}(J)$ and (up to replace J by one of its backward iterate), $f: J_{-n} \rightarrow J_{-n+1}$ is a homeomorphism for each $n \geq 0$. We now use the Denjoy-Schwartz argument to find a contradiction.

Let us fix $n \geq 0$. For each $0 \leq k \leq n$, the interval $f^k(J_{-n})$ is contained in one of the domains L_1 or L_2 . Hence $f^k|_{J_{-n}}$ coincides with a composition of the maps S_1 and S_2 and is associated to a return $s_k > 0$ of L . Note that $\tilde{P}_{-s_k}(J_{-n}) = J_{k-n}$ is a C^1 -curve in $R(L)$ tangent to the cone field \mathcal{C}^ε . By Proposition 3.40, there exists $\Delta > 0$ such that any sub-rectangle of $R(L)$, bounded by two leaves $W_{R(L)}^u(u), W_{R(L)}^u(u')$ has distortion bounded by Δ . Hence $\tilde{P}_{-s_k}(J_{-n})$ has length bounded by $\Delta \cdot |J_{k-n}|$. Consequently the sum $\sum_{k=0}^n |P_{-s_k}(J_{-n})|$ is uniformly bounded as n increases. The difference $s_{k+1} - s_k$ is uniformly bounded also, so that there exists a uniform bound C_{Sum} satisfying

$$\sum_{0 \leq m < s_n} |P_{-m}(J_{-n})| < C_{Sum}.$$

From Lemma 3.27, there exists an interval $\hat{J}_{-n} \subset \mathcal{W}^{cs}(x)$ containing J_{-n} satisfying $|\hat{J}_{-n}| \leq 2|J_{-n}|$ such that each component of $P_{-s_n}(\hat{J}_{-n} \setminus J_n)$ has length $\eta_S |P_{-s_n}(J_{-n})|$, where $\eta_S > 0$ is a small uniform constant. Note that the projection through unstable holonomy of $\tilde{P}_{-s_n}(\hat{J}_{-n})$ in L contains a uniform neighborhood $\hat{J} = S_{-s_n}(\hat{J}_{-n})$ of J .

The large integer n can be chosen so that the small interval J_{-n} is arbitrarily close to J . Consequently, \hat{J}_{-n} is contained in \hat{J} . This means $S_{-s_n}(\hat{J}_{-n}) \supset \hat{J}_{-n}$ implying that \hat{J}_{-n} contains a fixed point of the map S_{-s_n} . This is a contradiction since we have assumed that all the returns are shifting. As a consequence f is a minimal homeomorphism, which implies the lemma. \square

Proof of Proposition 4.21. Let us consider some intervals L_1, L_2, L and some returns t_1, t_2 as in Lemma 4.22. We denote $\tilde{P}_i = \tilde{P}_{-t_i}$ and $S_i = S_{-t_i}$, $i = 1, 2$. Since t_1, t_2 are large inside deep sequences of returns, the compositions $S_{i_k} \circ \dots \circ S_{i_1}$, when they are defined, are associated to returns of L (see Lemma 4.16).

Claim. \tilde{P}_1 and \tilde{P}_2 commute.

Proof. Both compositions $\tilde{P}_1 \circ \tilde{P}_2$ and $\tilde{P}_2 \circ \tilde{P}_1$ are associated to returns $s, s' > 0$. Note that the common endpoint of L_1 and L_2 has the same image by $S_1 \circ S_2$ and $S_2 \circ S_1$. If the two compositions do not coincide, the two returns are different, for instance $s' > s$, but there exists two points u, u' in $R(L)$ in a same unstable manifold such that $u = \pi_x \circ P_{-s'+s} \circ \pi_{\varphi_{-s}(x)}(u')$. Note that since t_1, t_2 can be obtained from deep sequences of returns, u, u' are arbitrarily close to L . The time $s' - s$ can be assumed to be arbitrarily large: otherwise, we let t_1, t_2 go to $+\infty$ in the deep sequences of returns; if $s' - s$ remain bounded, the points u, u' converge to a point of L a point which is fixed by some limit of the $\pi_x \circ P_{-s'+s}$. This is a contradiction since the returns of I are non-shifting. Since $s' - s$ is large, Lemma 4.14 implies that there is a large return sending u on u' , which is a contradiction since large returns are shifting. Consequently the two returns are the same and the compositions coincide. \square

By the properties on S_1, S_2 , there is $(i_k) \in \{1, 2\}^{\mathbb{N}}$ such that for each k

$$\tilde{P}_{i_k} \circ \dots \circ \tilde{P}_{i_1}(x) \in R(L).$$

From Lemma 4.16, for each k there exists a return s_k such that

$$\tilde{P}_{-s_k} = \tilde{P}_{i_k} \circ \cdots \circ \tilde{P}_{i_1}.$$

The dynamics of S_1, S_2 is minimal in L , hence the iterates $x_k := \varphi_{-s_k}(x)$ have a subsequence $x_{k(j)}$ converging to a point $x' \in K \setminus V$ such that $\pi_x(x')$ belongs to the unstable manifold of x . We define the intervals J, J_1, J_2 as limits of the iterates of L, L_1, L_2 by the maps $P_{-s_{k(j)}}$. In particular J is a δ -interval and the first item of Definition 4.20 holds.

By Remark 4.10, there exist times t'_1, t'_2 such that backward orbit of x during time $[-t_i, 0]$ is shadowed by the backward orbit of x' during the time $[-t'_i, 0]$. By the Global invariance, the maps $\pi_{x'} \circ P_{-t'_i}$, $i = 1, 2$, from a neighborhood of $0_{x'}$ to $\mathcal{N}_{x'}$ are conjugated to \tilde{P}_i by the projection π_x and will be still denoted by \tilde{P}_i . In particular, the map $P_{-s_{k(j)}}$ from a neighborhood of $0_x \in \mathcal{N}_x$ to $\mathcal{N}_{x'}$ coincides with $\tilde{P}_{i_{k(j)}} \circ \cdots \circ \tilde{P}_{i_1} \circ \pi_{x'}$ and $\pi_{x'} \circ \tilde{P}_{i_{k(j)}} \circ \cdots \circ \tilde{P}_{i_1}$.

Claim. J is aperiodic.

Proof. The projections by $\pi_{x'}$ of L_1, L_2 and $S_1(L_1), S_2(L_2)$ have images under the maps $P_{-s_{k(j)}}$ which converge to subintervals of J : by definition, the first ones are J_1, J_2 whereas by Global invariance the last ones denoted by J'_1, J'_2 have disjoint interior and satisfy $J = J'_1 \cup J'_2$. Since $S_1(L_1)$ and $\tilde{P}_1(L_1)$ have the same projection by unstable holonomy, Note that J'_1, J'_2 are also the limits of $\tilde{P}_1(L_1)$ and $\tilde{P}_2(L_2)$ under the maps $P_{-s_{k(j)}}$. Since \tilde{P}_1 and \tilde{P}_2 commute this implies that $J'_1 = \tilde{P}_1(J_1)$ and $J'_2 = \tilde{P}_2(J_2)$. Consequently the second item of the Definition 4.20 holds.

Let us consider the projection φ from L to J obtained as composition of $\pi_{x'}$ with the unstable holonomy. Note that it conjugates the orbit of 0_x in L by S_1, S_2 with the orbit of $0_{x'}$ in J by \tilde{P}_1, \tilde{P}_2 . Passing to the limit φ induces a conjugacy between the dynamics of S_1, S_2 on L and \tilde{P}_1, \tilde{P}_2 on J . The third item is thus a consequence of Lemma 4.22 \square

The proof of Proposition 4.21 is now complete. \square

4.5 Construction of a normally expanded irrational torus

The whole section is devoted to the proof of the next proposition.

Proposition 4.23. *If $x \in K \setminus V$ has an aperiodic δ -interval J , then the orbit of x is contained in a minimal set $\mathcal{T} \subset K$ which is a normally expanded irrational torus, and $\pi_x(\mathcal{T} \cap B(x, r_0/2)) \supset J$.*

Proof. The definition of aperiodic δ -interval is associated to two returns $t_1, t_2 > 0$. As before we denote $\tilde{P}_1 = \tilde{P}_{-t_1}$ and $\tilde{P}_2 = \tilde{P}_{-t_2}$.

Let us define \mathcal{C} to be the compact set of points $z \in K$ such that

$$d(z, x) \leq r_0/2 \text{ and } \pi_x(z) \in J.$$

The map $z \mapsto \pi_x(z)$ is continuous from \mathcal{C} to J . We also define the invariant set \mathcal{T} of points whose orbit meets \mathcal{C} .

For any $u \in J$, we define the sets

$$\Lambda_u^+ = \{v \in J, \exists n \in \mathbb{N}, k(1), \dots, k(n) \in \{1, 2\}, \text{ s.t. } \tilde{P}_{k(n)} \circ \tilde{P}_{k(n-1)} \circ \tilde{P}_{k(1)} u = v\},$$

$$\Lambda_u^- = \{v \in J, \exists n \in \mathbb{N}, k(1), \dots, k(n) \in \{1, 2\}, \text{ s.t. } \tilde{P}_{k(n)}^{-1} \circ \tilde{P}_{k(n-1)}^{-1} \circ \tilde{P}_{k(1)}^{-1} u = v\}.$$

By the definition of aperiodic interval and of Λ_u^+ and Λ_u^- , the set of accumulation points of Λ_u^+ and Λ_u^- are J .

Claim 4. *The map $z \mapsto \pi_x(z)$ from \mathcal{C} to J is surjective.*

Proof. This follows from the fact that the closure of $\Lambda_{0_x}^+$ is J , and that $\tilde{P}_{k(n)} \circ \tilde{P}_{k(n-1)} \circ \tilde{P}_{k(1)} \cdot 0_x$ belongs to the projection of the orbit of x by π_x . \square

Claim 5. *For any non-trivial interval $J' \subset J$ there is $T > 0$ such that any $z \in \mathcal{C}$ has iterates $\varphi_{t^+}(z)$, $\varphi_{-t^-}(z)$ with $t^+, t^- \in (1, T)$ which belong to \mathcal{C} and project by π_x in J' .*

Proof. For $z \in \mathcal{C}$, since Λ_z^+ is dense in J , there are $u_0 = \pi_x(z)$, $u_1, \dots, u_{\ell(z)}$ in J such that

- $u_{\ell(z)}$ belongs to the interior of J' ,
- For $0 \leq i \leq \ell(z) - 1$, there is $k(i) \in \{1, 2\}$ such that $\tilde{P}_{k(i)}(u_i) = u_{i+1}$.

Thus by compactness, there is a uniform ℓ such that $\ell(z) \leq \ell$ for any $z \in \mathcal{C}$. Moreover by the Global invariance, there is $t(i) > 0$ such that $\pi_x \varphi_{-t(i)}(z_i)$ is close to u_i and $t(i+1) - t(i)$ is smaller than $2 \max(t_1, t_2)$.

Since $\pi_x(z) \in J$ there exists a 2-Lipschitz homeomorphism θ of $[0, +\infty)$ such that $\varphi_{-\theta(t)}(z)$ is close to $\varphi_{-t}(x)$ for each $t \geq 0$. This proves that $\varphi_{-\theta(t(\ell))}(z)$ projects by π_x on u_ℓ , hence belongs to \mathcal{C} . Since ℓ and the differences $t(i+1) - t(i)$ are bounded and since θ is 2-Lipschitz, the time $t^- := \theta(t(\ell))$ is bounded by a constant which only depends on J' , as required.

The time t^+ is obtained in a similar way since Λ_z^- is dense in J for any $z \in J$. \square

Set $\mathcal{T} = \cup_{t \in \mathbb{R}} \varphi_t(\mathcal{C})$. From the previous claim, there exists $T > 0$ such that $\mathcal{T} = \varphi_{[0, T]}(\mathcal{C})$. Since \mathcal{C} is compact, \mathcal{T} is also compact. Note that (using Claim 5) any point in \mathcal{T} has arbitrarily large forward iterates in \mathcal{C} , whose projection by π_x belongs to $J \subset \mathcal{N}_x$. Since $x \in K \setminus V$, by choosing $\delta > 0$ small enough, the local injectivity implies:

Claim 6. *Any point in \mathcal{T} has arbitrarily large forward iterates in the r_0 -neighborhood of $K \setminus V$.*

Claim 7. *There exists a curve $\bar{\gamma} \subset \mathcal{C} \cap B(x, r_0/2)$ which projects homeomorphically by $z \mapsto \pi_x(z)$ on a non trivial compact interval of $\text{Interior}(J)$.*

Proof. We note that:

1. By the “No small period” assumption, for $k \in \mathbb{N}$, there exists $\varkappa_k > 0$ such that for any $y, y' \in \mathcal{C}$ and $t \in [0, 1]$ satisfying $d(y, y') < \varkappa_k$ and $d(y, \varphi_t(y')) < \varkappa_k$, then for any $s \in [0, t]$ we have $d(y, \varphi_s(y')) < 2^{-k-1}r_0$.
2. For each $\varkappa_k > 0$ as in Item 1, there exists $\varepsilon_k > 0$ with the following property. For any $y \in \mathcal{C}$ satisfying $B(y, \varkappa_k) \subset B(x, r_0)$ and for any $u \in J$ such that $d(u, \pi_x(y)) < \varepsilon_k$, then there is $y' \in B(y, \varkappa_k/2) \cap \mathcal{C}$ such that $\pi_x(y') = u$. Indeed, by the Local injectivity property, there is $\beta_k > 0$ such that for any points $w \in B(y, r_0)$, if $\|\pi_y w\| < \beta_k$, then there is $t \in [-1/2, 1/2]$ such that $d(\varphi_t(w), y) < \varkappa_k/2$. By the uniform continuity of the identification π , for $\varepsilon_k > 0$ such that for any $v_1, v_2 \in \mathcal{N}_x$, if $\|v_1 - v_2\| < \varepsilon_k$, then $\|\pi_y(v_1) - \pi_y(v_2)\| < \beta_k$. Now for any $u \in J$ such that $d(u, \pi_x(y)) < \varepsilon_k$, by Claim 4 there is $y_0 \in B(x, r_0/2) \cap \mathcal{C}$ such that $\pi_x(y_0) = u$ and $d(\pi_x(y_0), \pi_x(y)) < \varepsilon_k$. Hence $d(\pi_y(y_0), 0_y) = d(\pi_y \circ \pi_x(y_0), \pi_y \circ \pi_x(y)) < \beta_k$. By using the Local injectivity property, $d(\varphi_t(y_0), y) < \varkappa_k/2$ for some $t \in [-1/2, 1/2]$. Set $y' = \varphi_t(y_0)$. By Local invariance, we have that $\pi_x(y') = u$.
3. By letting $k \rightarrow \infty$ and $\beta_k \rightarrow 0$, one deduces that for any $u \in J$ and any $y \in \mathcal{C}$ such that $\pi_x(y)$ is close to u , there exists y' close to y such that $\pi_x(y') = u$.

We build inductively an increasing sequence of finite sets Y_k in \mathcal{C} satisfying:

- π_x is injective on Y_k ,

- for any $y, y' \in Y_k$ such that $\pi_x(y), \pi_x(y')$ are consecutive points of $\pi_x(Y_k)$ in J ,

$$d(\pi_x(y), \pi_x(y')) < \varepsilon_k.$$

- if moreover $y'' \in Y_{k+1}$ satisfies that $\pi_x(y'')$ is between $\pi_x(y)$ and $\pi_x(y')$ in J , then y'' is $2^{-k}r_0$ -close to y and y' .

Let us explain how to obtain Y_{k+1} from Y_k . We fix $y, y' \in Y_k$ such that $\pi_x(y), \pi_x(y')$ are consecutive points of $\pi_x(Y_k)$ in J and define the points in Y_{k+1} which project between $\pi_x(y), \pi_x(y')$. By items 2, we introduce a finite set $y''_1, y''_2, \dots, y''_j$ of points in $B(y, \varkappa_k/2)$ such that

- $y''_1 = y$ and $y''_j = y'$,
- $\pi_x\{y''_1, y''_2, \dots, y''_j\}$ is ε_{k+1} -dense in the arc of J bounded by $\pi_x(y)$ and $\pi_x(y')$,
- $\pi_x(y''_i), \pi_x(y''_{i+1})$ are consecutive points of the projections in J for $i \in \{1, \dots, j-1\}$,

The distance of y''_i and y''_{i+1} may be larger than $r_0 2^{-k-2}$. We need to modify the construction. By Item 2, there is $t_i \in [-1, 1]$ such that $z_i = \varphi_{t_i}(y''_i)$ is $\varkappa_{k+1}/2$ -close to y''_{i+1} . Choose $n \in \mathbb{N}$ large enough. Consider the points $\{\varphi_{mt_i/n}(y''_i)\}_{m=0}^n$. By using the item 3 there exists a finite collection X_i of points that are arbitrarily close to the set $\{\varphi_{mt_i/n}(y''_i)\}_{m=0}^n$ such that they have distinct projection by π_x and such that any two such point with consecutive projections are $2^{-k-2}r_0$ -close. By the item 1, the set X_i is contained in $B(y, 2^{-k-1}r_0)$. Since $d(y, y') < 2^{-k-1}r_0$, it is also contained in $B(y', 2^{-k}r_0)$. The set of points of Y_{k+1} which project between $\pi_x(y), \pi_x(y')$ is the union of the $\{y''_i, y''_{i+1}\} \cup X_i$ for any i .

Let us define $Y = \cup Y_k$. The restriction of π_x to Y is injective and has a dense image in a non-trivial interval J' contained in $\text{Interior}(J)$. Its inverse χ is uniformly continuous: indeed for any k , any $y, y' \in Y_k$ with consecutive projection, and any $y'' \in Y$ such that $\pi_x(y'')$ is between $\pi_x(y), \pi_x(y')$, the distance $d(y'', y)$ is smaller than $2^{-k+1}r_0$. As a consequence χ extends continuously to J' as a homeomorphism such that $\pi_x \circ \chi = \text{Id}_{J'}$. The curve $\bar{\gamma}$ is the image of χ . \square

Let γ be the open curve obtained by removing the endpoints of $\bar{\gamma}$. By the Local invariance, for $\varepsilon > 0$ small, φ is injective on $[-\varepsilon, \varepsilon] \times \bar{\gamma}$; its image is contained in \mathcal{C} and is homeomorphic to the ball $[0, 1]^2$. This is because a continuous bijective map from a compact space to a Hausdorff space is a homeomorphism. Thus φ is a homeomorphism from $(-\varepsilon, \varepsilon) \times \gamma$ to its image.

Claim 8. *The set $\varphi_{(-\varepsilon, \varepsilon)}(\gamma)$ is open in \mathcal{T} .*

Proof. Let us fix $z_0 \in \varphi_t(\gamma)$ for some $t \in (-\varepsilon, \varepsilon)$ and let us consider any point $z \in \mathcal{T}$ close to z_0 . We have to prove that z belongs to the open set $\varphi_{(-\varepsilon, \varepsilon)}(\gamma)$. Note that $d(z, x) < r_0$ and $\pi_x(z)$ belongs to $R(J)$. By Claim 5 and the definition of \mathcal{T} , the point z has a negative iterate $z' = \varphi_{-s}(z)$ in \mathcal{C} , with s bounded, so that $\pi_x(z') \in J$.

One deduces that $\pi_x(z)$ belongs to J . Otherwise, $\pi_x(z) \in R(J) \setminus J$. Since z and x have arbitrarily large returns in the r_0 -neighborhood of $K \setminus V$ (Claim 6), the Proposition 3.25 can be applied and $\pi_x \circ P_s \circ \pi_{z'}(J)$ is disjoint from J . Since s is bounded, the distance $d(\pi_x \circ P_s \circ \pi_{z'}(J), J)$ is bounded away from zero. We have $\pi_x(z_0) \in J$ whereas $\pi_x(z) \in P_s \circ \pi_{z'}(J)$ which contradicts the fact that z is arbitrarily close to z_0 .

Since z is close to z_0 and the curve γ is open, we have $\pi_x(z) \in \pi_x(\gamma)$ and $z \in \varphi_{t'}(\gamma)$ for some $|t'| < 1/2$ by the Local injectivity.

By the Local invariance $\varphi_{-t}(z)$ and $\varphi_{-t'}(z)$ have the same projection by π_x . If z is close enough to z_0 , both are arbitrarily close to γ . Since π_x is injective on γ one deduces that $d(\varphi_{-t}(z), \varphi_{-t'}(z))$ is arbitrarily small. By the “No small period” assumption, this implies that $|t' - t|$ is arbitrarily small. Hence $|t'| < \varepsilon$ proving that $z \in \varphi_{(-\varepsilon, \varepsilon)}(\gamma)$, as required. \square

By Claim 5, any point $z \in \mathcal{T}$ has a backward iterate in \mathcal{C} which project by π_x in $\pi_x(\gamma)$, hence by Local injectivity has an iterate $\varphi_{-t}(z)$ in γ . Since φ_{-t} is a homeomorphism, one deduces that z has an open neighborhood of the form $\varphi_t(\varphi_{(-\varepsilon, \varepsilon)}(\gamma))$ which is homeomorphic to the ball $(0, 1)^2$. As a consequence, \mathcal{T} is a compact topological surface.

Since any forward and backward orbit of \mathcal{T} meets the small open set $\varphi_{(-\varepsilon', \varepsilon')}(\gamma')$, for any $\varepsilon' \in (0, \varepsilon)$ and $\gamma' \subset \gamma$, the dynamics induced by (φ_t) on \mathcal{T} is minimal.

From classical results on foliations on surfaces (see [HH, Theorem 4.1.10, chapter I]), we get:

Claim 9. *\mathcal{T} is homeomorphic to the torus \mathbb{T}^2 and the induced dynamics of $(\varphi_t)_{t \in \mathbb{R}}$ on \mathcal{T} is topologically conjugated to the suspension of an irrational rotation of the circle.*

By Claim 8, $\varphi_{(-\varepsilon, \varepsilon)}(\gamma)$ is a neighborhood of x in \mathcal{T} and $\pi_x(\gamma)$ is contained in $\mathcal{W}^{cs}(x)$, hence is a C^1 -curve tangent to $\mathcal{E}(x)$ at 0_x . Hence \mathcal{T} is a normally expanded irrational torus. Moreover $\pi_x(\mathcal{T} \cap B(x, r_0/2))$ contains J by Claim 4. This ends the proof of Proposition 4.23. \square

4.6 Proof of the topological stability

We prove Proposition 4.5 and Lemma 4.4 that imply the topological stability in Section 4.1.2.

Proof of Proposition 4.5. By Proposition 4.7, there is a sequence of δ -intervals (\hat{I}_k) at $T_{\mathcal{F}}$ -Pliss backward iterates $\varphi_{-n_k}(x)$ in $K \setminus V$ and converging to a δ -interval I_∞ at a $T_{\mathcal{F}}$ -Pliss point x_∞ .

Let us assume first that all large returns of I_∞ are shifting. By Proposition 4.19, I_∞ has two deep sequences of returns, one shifting to the right and one shifting to the left. Proposition 4.21 then implies that $\alpha(x_\infty)$ contains a point $x' \in K \setminus V$ with an aperiodic δ -interval. By Proposition 4.23, the orbit of x' is contained in a normally expanded irrational torus \mathcal{T} . We have proved that the first case of Proposition 4.5 holds.

Let us assume then that I_∞ admits arbitrarily large non-shifting returns. Proposition 4.18 implies that I is contained in the unstable set of some periodic δ -interval J . \square

Proof of Lemma 4.4. Let us consider the rectangle $R(I)$: it is mapped into itself by P_{-2s} , where s is the period of q . By assumption (A3), \mathcal{E} is contracted over the orbit of q . One deduces that there exists a neighborhood B of 0_q in $R(I)$ such that for any $u \in B \setminus \mathcal{W}^{cu}(0_q)$, the backward orbit of u by (P_t) converges towards a periodic point of I which is not contracting along \mathcal{E} . So Lemma 3.7 implies that $\pi_q(K)$ is disjoint from $B \setminus \mathcal{W}^{cu}(0_q)$.

For any interval $L \subset \mathcal{N}_z$ as in Lemma 4.4, the point $\pi_q(z)$ does not belong to $B \setminus \mathcal{W}^{cs}(0_q)$. Hence $\pi_q(L)$ contains an interval J which is contained in B , meets $\mathcal{W}^{cu}(0_q)$ and has a length larger than some constant $\chi_0 > 0$. The backward iterate $P_{-2s}(J)$ still contain an interval J' having this property since q is attracting along \mathcal{E} .

Since $\pi_q(J)$ intersects $R(I)$, there exists $\theta \in \text{Lip}$ such that $d(\varphi_{-t}(z), \varphi_{-\theta(t)}(q))$ remain small for any $t \in [0, T]$. By the Global invariance, if k is the largest integer such that $\theta(T) > ks$, then $\varphi_{-\theta^{-1}(ks)}(L)$ contains an interval of length larger than $\chi_0/2$. Since $T_\theta^{-1}(T)$ is bounded, this implies that $\varphi_{-T}(L)$ has length larger than some constant $\chi > 0$. \square

4.7 Proof of the topological contraction

We prove a proposition that will allow us to conclude the topological contraction.

Definition 4.24. *K admits arbitrarily small periodic intervals if for any $\delta > 0$, there is a periodic point $p \in K$, whose orbit supports a periodic δ -interval.*

Proposition 4.25. *Under the assumptions of Theorem 4.1, if K does not contain a normally expanded irrational torus, if it admits arbitrarily small periodic intervals and is transitive, then there are $C_0 > 0$, $\varepsilon_0 > 0$, and a non-empty open set $U_0 \subset K$ such that for any $z \in U_0$, we have*

$$\sum_{k \in \mathbb{N}} |P_k(\mathcal{W}_{\varepsilon_0}^{cs}(z))| < C_0.$$

In the next 3 sections, we assume that the setting of Proposition 4.25 holds. We also assume that \mathcal{E} is not uniformly contracted (since otherwise Proposition 4.25 holds by Lemma 3.30). Proposition 4.25 is proved in Section 4.7.3, and then Theorem 4.1 is proved in Section 4.7.4.

4.7.1 The unstable set of periodic points

Lemma 4.26. *For any $\beta > 0$, there exist:*

- a periodic point $p \in K \setminus V$ (with period T),
- a point $x \in K \setminus \{p\}$ which is $r_0/2$ -close to p and whose α -limit set is the orbit of p ,
- $r_x > 0$ and a connected component Q of $B(0_x, r_x) \setminus \mathcal{W}^{cu}(x)$ in \mathcal{N}_x such that $Q \cap \pi_x(K) = \emptyset$ and the diameter of $P_{-t}(Q)$ is smaller than β for each $t > 0$.

Proof. Since K admits arbitrarily small periodic intervals, there is a periodic point $p \in K$ with period $T > 0$ and a periodic δ -interval $I \subset \mathcal{N}_p$ for δ small. Since \mathcal{E} is uniformly contracted in $V \supset K \setminus U$ (assumption (A2)), one can replace p by one of its iterates so that $p \in K \setminus V$. By Lemma 4.8 (and the continuity of (P_t)),

- the restriction of \mathcal{F} to the orbit of I by (P_t) is an expanded bundle.

Since \mathcal{E} is uniformly contracted over the orbit of 0_p (by (A3)), one can assume that:

- Only the endpoints of I are fixed by P_T . One is 0_p , the other one attracts any point of $I \setminus \{0_p\}$ by negative iterations of P_T .
- There is $\beta_p > 0$ such that for any $u \in \mathcal{N}_p$ satisfying $\|P_{-t}(u)\| < \beta_p$ for each $t > 0$, then u is in the unstable manifold of 0_p for P_t .

Let $r_p > 0$ such that $B(p, r_p) \subset U$ and $P_{-t} \circ \pi_p(B(p, r_p)) \subset B(0_{\varphi_{-t}(p)}, \beta_p)$ for each $t \in [0, T]$.

Since K is transitive, there are sequences (x_n) in K and (t_n) in $(0, +\infty)$ such that

- $\lim_{n \rightarrow \infty} x_n = p$ and $\lim_{n \rightarrow \infty} t_n = +\infty$,
- $d(\varphi_t(x_n), \text{Orb}(p)) < r_p$ for $t \in (0, t_n)$ and $d(\varphi_{t_n}(x_n), \text{Orb}(p)) = r_p$ for each n .

Taking a subsequence if necessary, we let $x := \lim \varphi_{t_n}(x_n)$. By the Global invariance, $P_{-t} \circ \pi_p(x) \subset B(0_{\varphi_{-t}(p)}, \beta_p)$ for each $t > 0$, hence $\pi_p(x)$ lies in the unstable manifold of 0_p . Combined with the Local injectivity, there exists $\theta \in \text{Lip}$ such that $\theta(0) = 0$ and $d(\varphi_{\theta(t)}(x), \varphi_t(p)) \rightarrow 0$ as $t \rightarrow -\infty$. Hence the α -limit set of x is $\text{Orb}(p)$.

We now consider the dynamics of P_T in restriction to \mathcal{N}_p . The periodic interval I is normally expanded. Consequently, for r_x small, one of the components Q of $B(0_x, r_x) \setminus \mathcal{W}^{cu}(x)$ in \mathcal{N}_x has an image by π_x contained in the unstable set of $I \setminus \{0_p\}$.

Let us assume by contradiction that there exists $y \in U$ such that $\pi_x(y) \in Q$. The backward orbit of $\pi_p(y)$ by P_T converges to the endpoint v of I which is not attracting along \mathcal{E} . Lemma 3.7 and the Global invariance imply that the backward orbit of y converges to a periodic orbit in K

whose eigenvalues at the period for the fibered flow coincide with the eigenvalues at the period of the fixed point v for P_T . This is a contradiction since all the eigenvalues of v are non-negative whereas \mathcal{E} is uniformly contracted over the periodic orbits of K . So Q is disjoint from $\pi_x(K)$.

One can choose p and I such that any iterates $P_{-t}(I)$ has diameter smaller than $\beta/2$. Reducing r_x , one can ensure that the iterates of $P_{-t}(Q)$ have diameter smaller than β until some time, where it stays close to the orbit of I and has diameter smaller than $2 \sup_t \text{Diam}(P_{-t}(I))$. This concludes the proof of the lemma. \square

4.7.2 Wandering rectangles

One chooses $\delta \in (0, r_0/2)$, $\beta, \varepsilon > 0$, and a component Q as in Lemma 4.26 such that

- if $\varphi_{-t}(x)$, $t > 1$, is 2δ -close to x , then $Q \cap \pi_x(P_{-t}(Q)) = \emptyset$;
- if $y, z \in K$ are close to x and $\theta \in \text{Lip}$ satisfies $d(\varphi_{\theta(t)}(y), \varphi_t(z)) < \delta$ for $t \in [-2, 0]$, then $\theta(0) - \theta(-2) > 3/2$;
- $\beta > 0$ associated to a shadowing at scale δ as in the Global invariance (Remarks 3.5.e);
- for any point z , the forward iterates of $\mathcal{W}_\varepsilon^{cs}(z)$ have length smaller than $\beta/3$ and than the constant δ_E given by Lemma 3.30 (this is possible since \mathcal{E} is topologically stable);
- the backward iterates of Q have diameter smaller than $\beta/2$.

For z close to x , one considers the closed curve $J(z) \subset \mathcal{W}^{cs}(z)$ of length ε bounded by 0_z , such that $\pi_x(J(z))$ intersects Q . Since $\pi_x(z)$ does not belong to Q (by Lemma 4.26), the unstable manifold of 0_p intersects $\pi_p(J(z))$, defining two disjoint arcs $J(z) = J^0(z) \cup J^1(z)$ such that $J^1(z)$ is bounded by 0_z , disjoint from Q and $\pi_x(J^0(z)) \subset Q$.

Let $H(z)$ denote the set of integers $n \geq 0$ such that $\varphi_n(z) \in K \setminus V$ and $(z, \varphi_n(z))$ is $T_{\mathcal{F}}$ -Pliss. For $n \in H(z)$, we set $J_n(z) = P_n(J(z))$ and $J_n^0(z) = P_n(J^0(z))$.

Lemma 4.27. *There is $C_R > 0$ with the following property. For any z close to x and any $n \in H(z)$, there exists a rectangle $R_n(z) \subset \mathcal{N}_{\varphi_n(z)}$ which is the image of $[0, 1] \times B_{d-1}(0, 1)$ by a homeomorphism ψ_n such that (see Figure 4):*

1. *Volume $(R_n(z)) > C_R \cdot |J_n^0(z)|$,*
2. *the preimages $P_{-t}(R_n(z))$ for $t \in [0, n]$ have diameter smaller than $\beta/2$,*
3. *$\pi_x \circ P_{-n}(R_n(z))$ is contained in Q .*

Proof. The construction is very similar to the proof of Proposition 4.9. Let us fix $\alpha' > 0$ and $\alpha_{min} > 0$ much smaller. One considers a rectangle in \mathcal{N}_z whose interior projects by π_x in Q , given by a parametrization ψ_0 such that $\psi_0([0, 1] \times \{0\}) = J^0(z)$ and each disc $\psi_0(\{u\} \times B_{d-1})$ is tangent to the center-unstable cones, has diameter smaller than α' and contains a center-unstable ball centered at $J^0(z)$ and with radius much larger than α_{min} . Moreover $\pi_p \circ \psi_0(\{0_p\} \times B_{d-1})$ is contained in the unstable manifold of p .

Since the center-unstable cone-field is invariant, at any iterate $\varphi_n(z)$ such that $(z, \varphi_n(z))$ is $T_{\mathcal{F}}$ -Pliss, one can build a similar rectangle R_n such that $P_{-t}(R_n)$, $0 < t < n$, has center-unstable disc of diameter smaller than α' and $P_{-n}(R_n) \subset R_0$.

Since the forward iterates of $J^0(z)$ have length smaller than $\beta/3$, by choosing α' small enough, one guaranties that the $P_{-t}(R_n)$, $0 < t < n$, have diameter smaller than $\beta/2$, and that $P_{-n}(R_n) \subset R_0(z)$ are contained in $\pi_z(Q)$. The estimate on the volume is obtained from Fubini theorem and the distortion estimate given by Proposition 3.40. \square

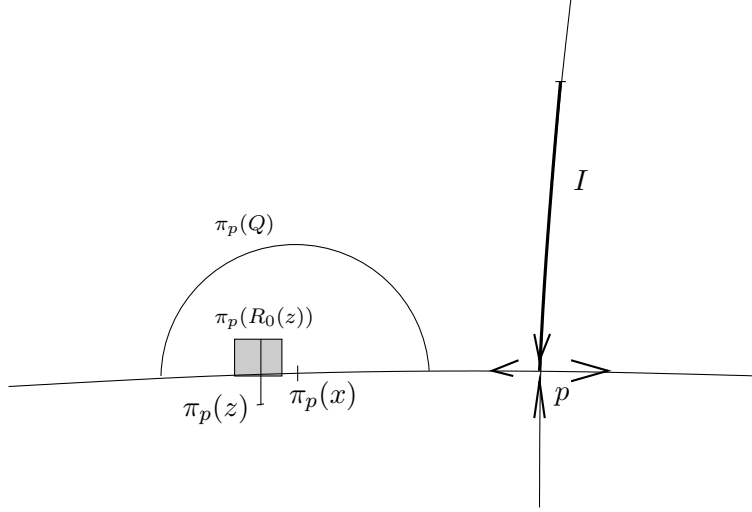


Figure 4

Lemma 4.28. *For each z close to x and each $n < m$ in $H(z)$, if $d(\varphi_n(z), \varphi_m(z)) < \delta$, then $\pi_{\varphi_n(z)}(R_m(z)) \cap R_n(z) = \emptyset$.*

Proof. Let us assume by contradiction that $\pi_{\varphi_m(z)}(R_m(z))$ and $R_n(z)$ intersect.

Since $P_{-t}(R_n(z) \cup J_n(z))$, $t \in [0, n]$, and $P_{-t}(R_m(z) \cup J_m(z))$, $t \in [0, m]$, have diameter smaller than β , the Global invariance (Remark 3.5.(e)) applies: there is $\theta \in \text{Lip}$ such that

- $|\theta(n) - m| < 1/4$,
- for any $t \in [0, n]$, one has $d(\varphi_t(z), \varphi_{\theta(t)}(z)) < \delta$,
- $\pi_x \circ P_{\theta(0)-m}(R_m(z))$ intersects $\pi_x \circ P_{-n}(R_n(z))$, hence Q .

In particular $\theta(n) > n + 1/2$ and Proposition 3.6 gives $\theta(0) > 2$.

Since the backward iterates of Q by P_t have diameter smaller than β , the Global invariance (item (e) in Remarks 3.5) can be applied to the points x and $\varphi_{\theta(0)}(z)$. It gives $\theta' \in \text{Lip}$ with $|\theta'(\theta(0))| \leq 1/4$ such that $d(\varphi_{\theta'(t)}(x), \varphi_t(z)) < \delta$ for each $t \in [0, \theta(0)]$. Moreover $\pi_x \circ P_{\theta'(0)}(Q)$ intersects $R_0(z)$, hence Q . We have $d(\varphi_{\theta'(0)}(x), x) < 2\delta$ and $1/4 > \theta'(\theta(0)) > \theta'(0) + 3/2$ by our choice of δ at the beginning of Section 4.7.2

We proved that $\pi_x \circ P_{-t}(Q) \cap Q \neq \emptyset$ for some $t > 1$ such that $d(\varphi_{-t}(x), x) < 2\delta$. This contradicts the choice of Q . The rectangles $\pi_{\varphi_n(z)}(R_m(z))$ and $R_n(z)$ are thus disjoint. \square

As a consequence of Lemma 4.28, one gets

Corollary 4.29. *There exists $C_H > 0$ such that for any z close to x ,*

$$\sum_{n \in H(z)} |J_n^0(z)| < C_H.$$

Proof. As in the proof of Lemma 4.11, one fixes a finite set $Z \subset U$ such that any point $z \in U$ is $\delta/2$ -close to a point of Z . Let C_{Vol} be a bound on the volume of the balls $B(0_z, \beta_0) \subset \mathcal{N}_z$ over $z \in K$ and let $C_H = 2C_R^{-1}C_{Vol}\text{Card}(Z)$. Since identifications are C^1 , up to reduce r_0 , one can assume that the modulus of their Jacobian is smaller than 2.

The statement now follows from the item 1 of Lemma 4.27 and the disjointness of the rectangles (Lemma 4.28). \square

4.7.3 Summability. Proof of Proposition 4.25

Lemma 4.30. *There exists $C_{sum} > 0$ such that for any point z close to x , we have*

$$\forall n \geq 0, \quad \sum_{k=0}^n |P_k(J^0(z))| < C_{sum}.$$

Proof. Let us denote by $n_0 = 0 < n_1 < n_2 < \dots$ the integers in $H(z)$. By Proposition 3.34, for any i , the piece of orbit $(\varphi_{n_i}(z), \varphi_{n_{i+1}}(z))$ is $(C_{\mathcal{E}}, \lambda_{\mathcal{E}})$ -hyperbolic for \mathcal{E} .

Let $\delta_E, C'_{\mathcal{E}}$ be the constants associated to $C_{\mathcal{E}}, \lambda_{\mathcal{E}}$ by Lemma 3.30. We have built $J(z)$ such that any forward iterate has length smaller than δ_E . Hence Lemma 3.30 implies that

$$\sum_{k=n_i}^{n_{i+1}} |P_k(J^0(z))| < C'_{\mathcal{E}} |J_{n_{i+1}}^0|.$$

With Corollary 4.29, one deduces

$$\sum_{k=0}^n |P_k(J^0(z))| < C_{sum} := C'_{\mathcal{E}} C_H.$$

□

We can now end the proof of the proposition.

Proof of the Proposition 4.25. Let $\eta_S > 0$ be the constant associated to C_{sum} by Lemma 3.27. If z belongs to a small neighborhood U_0 of x and if ε_0 is small enough, the intervals $\mathcal{W}_{\varepsilon_0}^{cs}(z)$ and $J^0(z)$ are both contained in an interval $\hat{J}(z) \subset \mathcal{W}^{cs}(z)$ such that $|\hat{J}(z)| \leq (1 + \eta_S)|J^0(z)|$. Combining Lemma 3.27 with Lemma 4.30, one gets

$$\sum_{k=0}^n |P_k(\mathcal{W}_{\varepsilon_0}^{cs}(z))| \leq \sum_{k=0}^n |P_k(\hat{J}(z))| < 2 \sum_{k=0}^n |P_k(J^0(z))| < C_0 := 2C_{sum}.$$

□

4.7.4 Proof of Theorem 4.1

We consider the set K as in the Theorem 4.1, and we assume by contradiction that none of the three properties in the statement of the theorem holds. In particular K does not contain a normally expanded irrational torus.

Claim. *K is transitive.*

Proof. Since \mathcal{E} is not uniformly contracted, there exists an ergodic measure whose Lyapunov exponent along \mathcal{E} is non-negative. Since \mathcal{E} is uniformly contracted on each proper invariant subset of K , the support of the measure coincides with K . Hence K is transitive. □

Let us fix $\delta > 0$ arbitrarily small. Since \mathcal{E} is topological stable, there is $\varepsilon > 0$ such that for any $x \in K$ and any $t > 0$, one has

$$P_t(\mathcal{W}_{\varepsilon}^{cs}(x)) \subset \mathcal{W}_{\delta}^{cs}(\varphi_t(x)).$$

Since the topological contraction fails, there are (x_n) in K , $(t_n) \rightarrow +\infty$ and $\chi > 0$ such that

$$\chi < |P_{t_n}(\mathcal{W}_{\varepsilon}^{cs}(x_n))| \text{ and } |P_{t_n}(\mathcal{W}_{\varepsilon}^{cs}(x_n))| < \delta, \quad \forall t > 0.$$

Let $I = \lim_{n \rightarrow \infty} P_{t_n}(\mathcal{W}_\varepsilon^{cs}(x))$. It is a δ -interval and by Proposition 4.5, it is contained in the unstable set of a periodic δ -interval since K contains no normally expanded irrational tori. This proves that K admits arbitrarily small periodic intervals and Proposition 4.25 applies.

One gets a non-empty open set $U_0 \subset K$ such that at any $z \in U_0$ a summability holds in the \mathcal{E} direction. With Lemma 3.27, one deduces that for any $x \in U_0$,

$$\lim_{n \rightarrow \infty} \|DP_n|_{\mathcal{E}(x)}\| = 0.$$

Now for any $z \in K$,

- either there is $t > 0$ such that $\varphi_t(z) \in U_0$ and then $\lim_{n \rightarrow \infty} \|DP_n|_{\mathcal{E}(z)}\| = 0$;
- or the forward orbit of z does not meet U_0 , then \mathcal{E} is contracted on the proper invariant compact set $\omega(z)$ and we also have $\lim_{n \rightarrow \infty} \|DP_n|_{\mathcal{E}(z)}\| = 0$.

By using a compactness argument, one deduces that \mathcal{E} is uniformly contracted on K . This contradicts our assumptions on K . Theorem 4.1 is now proved. \square

5 Markovian boxes

We will build boxes with a Markovian property for C^2 local fibered flow having a dominated splitting $\mathcal{N} = \mathcal{E} \oplus \mathcal{F}$ with two-dimensional fibers, such that \mathcal{E} is topologically contracted.

Standing assumptions. Keep assumptions (A1), (A2), (A3) of Section 4 and add furthermore:

- (A4) \mathcal{E} is topologically contracted and \mathcal{F} is one-dimensional,
- (A5) there exists an ergodic measure μ for (φ_t) whose Lyapunov exponent along \mathcal{F} is positive, and whose support is not a periodic orbit and intersects $K \setminus \overline{V}$.

5.1 Existence of Markovian boxes

We fix a non-periodic point $x \in K \setminus \overline{V}$ in the support of μ . In particular taking r_0 small enough, the ball $U(x, r_0)$ centered at x and with radius r_0 in K is contained in U . We also denote $\mu_x = (\pi_x)_*(\mu|_{U(x, r_0)})$ and fix some $\beta_x > 0$.

Definition 5.1. A box $B \subset \mathcal{N}_x$ is the image by a homeomorphism ψ such that:

- $\partial^{\mathcal{F}} B := \psi(\{0, 1\} \times [0, 1])$ is C^1 , tangent to $\mathcal{C}^{\mathcal{F}}$ (and called the \mathcal{F} -boundary),
- $\partial^{\mathcal{E}} B := \psi([0, 1] \times \{0, 1\})$ is C^1 , tangent to $\mathcal{C}^{\mathcal{E}}$ (and called the \mathcal{E} -boundary).

A center-stable sub-box (resp. center-unstable sub-box) is a box $B' \subset B$ such that

$$\partial^{\mathcal{F}} B' \subset \partial^{\mathcal{F}} B \text{ (resp. } \partial^{\mathcal{E}} B' \subset \partial^{\mathcal{E}} B).$$

In particular a box is a rectangle as defined in Section 3.4.5.

Definition 5.2. Let us fix some constants $C_{\mathcal{F}}, \lambda_{\mathcal{F}} > 1$.

A transition between boxes $B, B' \subset \mathcal{N}_x$ is defined by some $y \in K$ and $t > 2$ such that:

- y and $\varphi_t(y)$ are $r_0/2$ -close to x ,
- $(y, \varphi_t(y))$ is $(C_{\mathcal{F}}, \lambda_{\mathcal{F}})$ -hyperbolic for \mathcal{F} ,
- y projects by π_x in the interior of B and $\varphi_t(y)$ in the interior of B' .

Two boxes B, B' have Markovian transitions if for any transition (y, t) between B and B' , there exist a center-stable sub-box $B^{cs} \subset B$ and a center-unstable sub-boxes $B^{cu} \subset B'$ whose interior contain $\pi_x(y)$ and $\pi_x(\varphi_t(y))$ respectively, such that $\pi_x P_t \pi_y(B^{cs}) = B^{cu}$.

The remainder of Section 5 is devoted to prove the following main result of this section.

Theorem 5.3 (Existence of Markovian boxes). *Under the assumptions above, there exists a box R in $B(0, \beta_x) \subset \mathcal{N}_x$ whose interior has positive μ_x -measure, such that for any $C_{\mathcal{F}}, \lambda_{\mathcal{F}} > 1$ (defining the transitions) and $\beta_{box} > 0$, the box R contains finitely many boxes B_1, \dots, B_k and $t_{box}, \Delta_{box} > 0$ satisfying the following properties:*

1. Any z which is $r_0/2$ -close to x and whose projection $\pi_x(z)$ belongs to $\partial^{\mathcal{E}} R$ (resp. to $\partial^{\mathcal{F}} R$) has a forward orbit (resp. a backward orbit) which accumulates on a periodic orbit of K .
2. The boxes B_1, \dots, B_k have disjoint interiors, their union contains $R \cap \pi_x(K)$, and their boundaries have zero measure for μ_x .
3. Any transition (y, t) with $t > t_{box}$ between any boxes B_i, B_j is Markovian.
4. The sub-boxes B^{cs}, B^{cu} associated to any transition (y, t) with $t > t_{box}$ between any B_i, B_j satisfy:

- $\text{Diam}(P_s \circ \pi_y(B^{cs})) < \beta_x$ for any $s \in (0, t)$,
- $\text{Diam}(P_s \circ \pi_y(B^{cs})) < \beta_{box}$ for any $s \in (t_{box}, t)$,
- B^{cu} has distortion bounded by Δ_{box} .

5. For any two transitions (y_1, t_1) and (y_2, t_2) with $t_1, t_2 > t_{box}$ such that the interiors of the sub-boxes B_1^{cu}, B_2^{cu} intersect, we have $B_1^{cu} \subset B_2^{cu}$ or $B_2^{cu} \subset B_1^{cu}$.

More precisely, $B_2^{cu} \subset B_1^{cu}$ holds if there exists $\theta \in \text{Lip}_2$ satisfying

- $\theta(t_1) \geq t_2 - 2$, $\theta^{-1}(t_2) \geq t_1 - 2$ and $\theta(0) \geq -1$,
- $d(\varphi_t(y_1), \varphi_{\theta(t)}(y_2)) < r_0/2$ for $t \in [0, t_1] \cap \theta^{-1}([0, t_2])$.

Up to exchange (y_1, t_1) and (y_2, t_2) , there exists such a θ satisfying $|\theta(t_1) - t_2| \leq 1/2$.

6. For any two transitions (y_1, t_1) and (y_2, t_2) with $t_1, t_2 > t_{box}$ such that $y_1 = y_2$ and $t_2 > t_1 + t_{box}$, we have $B_2^{cs} \subset B_1^{cs}$.

5.2 Construction of boxes

5.2.1 Notations, choices of constants

One will consider some small numbers $\alpha_0, \eta, \alpha_x, \beta_x > 0$, that are chosen in this order, according to the properties stated in this subsection. The constant α_0 will bound the size of the plaques. In the whole Section 5, one will work with generalized orbits $\bar{u} = (u(t))$ in the η -neighborhood of K . The number α_x controls the hyperbolicity inside the center-unstable plaques. At last, $\beta_x > 0$ is the constant introduced at the beginning of Section 5.1, that can be reduced to be much smaller than the other quantities.

The plaques $\mathcal{W}^{cs}, \mathcal{W}^{cu}$.

Lemma 5.4. *There exist center-stable and center-unstable plaques $\mathcal{W}^{cs}(\bar{u}), \mathcal{W}^{cu}(\bar{u})$ that have length smaller than α_0 , depend continuously on \bar{u} and satisfy moreover:*

- *The center-unstable plaques are locally invariant: there is $\alpha_{\mathcal{F}} \in (0, \alpha_0)$ such that*

$$\bar{P}_{-1}(\mathcal{W}_{\alpha_{\mathcal{F}}}^{cu}(\bar{u})) \subset \mathcal{W}^{cu}(\bar{P}_{-1}(\bar{u})).$$

- *The center-unstable plaques are coherent: the statement of Proposition 3.25 holds for \mathcal{W}^{cu} , the constants $\eta, \alpha_{\mathcal{F}}$ (and the flow (P_{-t})).*
- *The center-stable plaques are trapped for time $s > 0$:*

$$\forall s > 0, \quad \bar{P}_s(\overline{\mathcal{W}^{cs}(\bar{u})}) \subset \mathcal{W}^{cs}(\bar{P}_s(\bar{u})).$$

- *The center-stable plaques are coherent: let \bar{u}, \bar{v} be any generalized orbits with $u(0) \in \mathcal{N}_y, v(0) \in \mathcal{N}_{y'}$ such that y, y' are $r_0/2$ -close to x and the projection $(y(t))_{t \in \mathbb{R}}$ of \bar{u} in K has arbitrarily large positive iterates in the r_0 -neighborhood of $K \setminus V$; if $\pi_x(\mathcal{W}^{cs}(\bar{u}))$ and $\pi_x(\mathcal{W}^{cs}(\bar{v}))$ intersect, then they are contained in a same C^1 curve.*

Proof. The first item is given by Proposition 3.20. It gives also a locally-invariant center-stable plaque family $\mathcal{W}^{cs,0}$. The second item is obtained by Proposition 3.25 directly.

Since \mathcal{E} is topologically contracted over the 0-section, there exists $\varepsilon > 0$ and $T_* > 0$ such that $\bar{P}_s(\mathcal{W}_{\varepsilon}^{cs,0}(\bar{u})) \subset \mathcal{W}^{cs,0}(\bar{P}_s(\bar{u}))$ for any $s > 0$ and which is trapped for times $s \geq T_*$. Let us choose $b > 0$ small and define

$$\mathcal{W}^{cs}(\bar{u}) = \bigcup_{0 \leq t \leq T_*} \bar{P}_t(\mathcal{W}_{\varepsilon+bt}^{cs,0}(\bar{P}_{-t}(\bar{u}))).$$

These plaques are open sets in $\mathcal{W}^{cs,0}$ and we have to check the trapping property at any time $s > 0$. Note that it is enough to choose $s \in (0, T_*)$.

Let us consider $t \in [0, T_*]$. In the case $t \geq T_* - s$ we set $t' = s + t - T_*$ and we have

$$\bar{P}_s(\bar{P}_t(\mathcal{W}_{\varepsilon+bt}^{cs,0}(\bar{P}_{-t}(\bar{u})))) = \bar{P}_{t'} \circ \bar{P}_{T_*}(\mathcal{W}_{\varepsilon+bt}^{cs,0}(\bar{P}_{-t-s} \circ \bar{P}_s(\bar{u}))).$$

By the trapping property at time T_* , provided b has been chosen small enough, one has

$$\bar{P}_{T_*}(\mathcal{W}_{\varepsilon+bt}^{cs,0}(\bar{P}_{-t-s} \circ \bar{P}_s(\bar{u}))) \subset \mathcal{W}_{\varepsilon+bt'-bs}^{cs,0}(\bar{P}_{T_*-s-t} \circ \bar{P}_s(\bar{u})).$$

Hence $\bar{P}_s(\bar{P}_t(\mathcal{W}_{\varepsilon+bt}^{cs,0}(\bar{P}_{-t}(\bar{u}))))$ is contained in $\bar{P}_{t'}(\mathcal{W}_{\varepsilon+bt'-bs}^{cs,0}(\bar{P}_{-t'} \circ \bar{P}_s(\bar{u})))$.

In the case $t < T_* - s$, we set $t' = t + s > t$ and we have

$$\begin{aligned} \bar{P}_s(\bar{P}_t(\mathcal{W}_{\varepsilon+bt}^{cs,0}(\bar{P}_{-t}(\bar{u})))) &= \bar{P}_{s+t}(\mathcal{W}_{\varepsilon+bt}^{cs,0}(\bar{P}_{-(t+s)} \circ \bar{P}_s(\bar{u})))) \\ &= \bar{P}_{t'}(\mathcal{W}_{\varepsilon+bt'-bs}^{cs,0}(\bar{P}_{-t'} \circ \bar{P}_s(\bar{u})))) \end{aligned}$$

The closure of $\bigcup_{0 \leq t' \leq T_*} \bar{P}_{t'}(\mathcal{W}_{\varepsilon+bt'-bs}^{cs,0}(\bar{P}_{-t'} \circ \bar{P}_s(\bar{u}))))$ is contained in $\mathcal{W}^{cs}(\bar{P}_s(\bar{u}))$.

Combining these two cases, one thus gets the third item:

$$\forall s > 0, \quad \bar{P}_s(\text{Closure}(\mathcal{W}^{cs}(\bar{u}))) \subset \mathcal{W}^{cs}(\bar{P}_s(\bar{u})).$$

For the fourth item, one uses the Local injectivity: since the plaque are small, one can assume that y, y' are η -close. Then Proposition 3.25 applies and gives the coherence. \square

The constants C_x, λ_x . For $C_x, \lambda_x > 1$, we introduce the set \mathcal{H} of points $y \in K$ that are $(C_x/2, \lambda_x^2)$ -hyperbolic for \mathcal{F} . Since the Lyapunov exponent of μ along \mathcal{F} is positive, we can fix C_x, λ_x such that the following set has positive μ -measure, for any $\beta_x > 0$:

$$H_x := \{y \in \mathcal{H}, d(y, x) < r_0/2 \text{ and } \pi_x(y) \in B(0_x, \beta_x)\}.$$

The choice of β_x will be fixed later.

The constant α_x . Up to reduce $\eta > 0$, there exists $\alpha_x > 0$ (depending on C_x, λ_x) such that for any generalized orbit \bar{u} , if \bar{u} is (C_x, λ_x) -hyperbolic for \mathcal{F} , then the set $\bar{P}_{-t}(\mathcal{W}_{\alpha_x}^{cu}(\bar{u}))$ is defined for any $t > 0$, has a diameter smaller than $\min(\eta, \alpha_{\mathcal{F}})\lambda_x^{-t/2}$ (see Proposition 3.38) and is contained in $\mathcal{W}^{cu}(\bar{P}_{-t}(\bar{u}))$ (by invariance, Remark 3.24).

Up to reduce η, α_x , a stronger coherence for center-unstable plaques is satisfied:

Lemma 5.5. *Consider two generalized orbits \bar{u}, \bar{v} and $t \geq 0$ such that:*

- $u(0) \in \mathcal{N}_y, v(-t) \in \mathcal{N}_{y'}$ with y, y' $r_0/2$ -close to x , and the projection $(y(t))_{t \in \mathbb{R}}$ of \bar{u} in K has arbitrarily large negative iterates in the r_0 -neighborhood of $K \setminus V$,
- \bar{v} is (C_x, λ_x) -hyperbolic for \mathcal{F} and $\pi_x(u(0)) \in \pi_x(\bar{P}_{-t}(\mathcal{W}_{\alpha_x}^{cu}(\bar{v})))$.

Then $\pi_x(\bar{P}_{-t}(\mathcal{W}_{\alpha_x}^{cu}(\bar{v}))) \subset \pi_x(\mathcal{W}^{cu}(\bar{u}))$.

Proof. This is a direct consequence of Proposition 3.25 (and the second item of Lemma 5.4) applied to the sets $X = \{u(0)\}$ and $X' = \bar{P}_{-t}(\mathcal{W}_{\alpha_x}^{cu}(\bar{v}))$. Indeed, the hyperbolicity of \bar{v} ensures that the diameter of the sets $\bar{P}_{-s}(X')$ is smaller than $\alpha_{\mathcal{F}}$ for any $s \geq 0$. \square

The choice of η is fixed now. One can build generalized orbits as local product between generalized orbits and orbits of K . Up to reduce α_x and β_x , one gets:

Lemma 5.6. *For any generalized orbit \bar{p} contained in the $\eta/2$ -neighborhood of K , and for any $y \in K, t \geq 0$ satisfying*

- y and z are $r_0/2$ -close to x , where $z \in K$ is the point such that $p(-t) \in \mathcal{N}_z$,
- \bar{p} is (C_x, λ_x) -hyperbolic for \mathcal{F} and $p(-t-s) = P_{1-s}(p(-t-1))$ for any $s \in (0, 1]$,
- $\pi_x(\mathcal{W}_{\beta_x}^{cs}(y))$ and $\pi_x(\bar{P}_{-t}(\mathcal{W}_{\alpha_x}^{cu}(\bar{p})))$ intersect and $d(z, y) < \eta/2$,

then there exists a generalized orbit \bar{u} (in the η -neighborhood of K) satisfying:

- $u(s) \in \mathcal{W}^{cs}(\varphi_s(y))$ for every $s \geq 0$,
- $\pi_x(u(0)) \in \pi_x(\bar{P}_{-t}(\mathcal{W}_{\alpha_x}^{cu}(\bar{p})))$ and $u(-s) \in \bar{P}_{-t-s}(\mathcal{W}_{\alpha_x}^{cu}(\bar{p}))$ for $s > 0$.

Remark. In the case $d(z, y) < \eta/2$ does not hold, one can choose by the Local injectivity some $\tau \in [-1/4, 1/4]$ such that $d(\varphi_{\tau}(z), y) < \eta/2$. One can then define in a same way a generalized orbit which satisfies $u(-s) \in \bar{P}_{-t-s+\tau}(\mathcal{W}_{\alpha_x}^{cu}(\bar{p}))$ for $s > \max(0, \tau)$.

Proof. Define $u(0)$ to be the intersection point between $\mathcal{W}_{\beta_x}^{cs}(y)$ and $\pi_y(\bar{P}_{-t}(\mathcal{W}_{\alpha_x}^{cu}(\bar{p})))$. Since β_x is small, the topological contraction of \mathcal{E} implies that $|u(s)| < \eta$ for each $s \geq 0$, where $u(s) := P_s(u(0))$. The generalized flow \bar{P} associated to the generalized orbit \bar{p} (see Definition 3.13) allows to define $u(s) := \bar{P}_s(\pi_z(u(0)))$ for each $s < 0$. Note that $u(s)$ belongs to $\bar{P}_s(\mathcal{W}_{\alpha_x}^{cu}(\bar{p}))$, whose diameter is smaller than η for any $s < 0$, provided α_x is chosen small enough since \bar{p} is (C_x, λ_x) -hyperbolic for \mathcal{F} . Note also that the projection of $(u(s))_{s \in [-1, 0]}$ on K is continuous.

We have thus defined in this way a generalized orbit, whose projection on K coincides with the projection of \bar{p} for times $s < 0$ and with $\varphi_s(y)$ for times $s \geq 0$. In order to check that it is contained in the η -neighborhood of K , it remains to show that:

- $y \in K \setminus V$: this follows from our assumptions,
- the projection of $p(s)$ on K for $s < 0$ close to 0 is η -close to y : this follows from the fact that \bar{p} is in the $\eta/2$ -neighborhood of K and that $d(z, y) < \eta/2$.

□

The constant β_x . The constant β_x chosen for Theorem 5.3 can be reduced to be smaller than α_0, η, α_x and to satisfy (using the trapping):

- for any $y \in K$ and $t \geq 1/4$ such that $y, \varphi_t(y)$ are $r_0/2$ -close to x , the two components of $\pi_x(\mathcal{W}^{cs}(\varphi_t(y)) \setminus P_t(\mathcal{W}^{cs}(y)))$ have length larger than $2\beta_x$.

Other constants. Once R is constructed, one can choose other numbers $C_{\mathcal{F}}, \lambda_{\mathcal{F}}, \beta_{box}$ as in Theorem 5.3. Another constant $\alpha_{box} > 0$ (which is a relaxed analogue of α_x) will be introduced later in Section 5.2.4, depending on these choices.

5.2.2 A shadowing lemma

Proposition 5.7. *For any $\delta > 0$, there exist $r > 0$ and $T_0 \geq 1$ such that for any points $y, \varphi_T(y) \in H_x$ that are r -close with $T \geq T_0$, there exists $p \in \mathcal{N}_x$ such that*

- $P_s(\pi_y(p))$ is defined and is contained in $B(0_{\varphi_s(y)}, \delta) \subset \mathcal{N}_{\varphi_s(y)}$ for any $s \in (0, T)$,
- p is fixed by $\tilde{P}_T := \pi_x \circ P_T \circ \pi_y$.

We could give an argument which uses the domination $\mathcal{E} \oplus \mathcal{F}$, similar to the construction in [CP, Proposition 9.6]. We propose here a more topological proof.

Proof. Let us choose $\varepsilon > 0$ much smaller than δ .

Claim. *If T_0 is large enough, for any $y, \varphi_T(y) \in H_x$ with $T \geq T_0$, there exists a box $B \subset \mathcal{N}_y$ such that:*

- $P_s(B)$ is a box of $B(0, \delta)$ in $\mathcal{N}_{\varphi_s(y)}$ for any $s \in [0, T]$,
- B contains $\mathcal{W}_{\varepsilon}^{cs}(y)$ and $P_T(B)$ contains $\mathcal{W}_{\varepsilon}^{cu}(\varphi_T(y))$.

Proof. Since \mathcal{E} is topologically contracted, if ε has been chosen small, the iterates $P_s(\mathcal{W}_{\varepsilon}^{cs}(y))$ are smaller than $\delta/10$ for any $s \in [0, T]$. We can choose two disjoint arcs L_0^-, L_0^+ of length 1, tangent to $\mathcal{C}^{\mathcal{F}}$, centered at the endpoints of $\mathcal{W}_{\varepsilon}^{cs}(y)$ and disjoint from $\mathcal{W}_{\varepsilon}^{cu}(y)$. Let us consider $L^- \subset L_0^-$ (resp. $L^+ \subset L_0^+$) the maximal arc whose iterates by P_s , $s \in [0, T]$, remain at distance smaller than δ from $0_{\varphi_s(y)}$. Since $(y, \varphi_T(y))$ is $(C_x/2, \lambda_x^2)$ hyperbolic for \mathcal{F} , and since the endpoints of $P_s(\mathcal{W}_{\varepsilon}^{cs}(y))$ are close to $0_{\varphi_s(y)}$, we deduce that $P_T(L^-)$ and $P_T(L^+)$ have length larger than 10ε .

Let us note that $P_T(L^-)$ and $P_T(L^+)$ are disjoint from $\mathcal{W}_{\varepsilon}^{cu}(\varphi_T(y))$: otherwise L^- (or L^+) would intersect $P_{-T}(\mathcal{W}_{\varepsilon}^{cu}(\varphi_T(y)))$, but these three curves have a length arbitrarily small if T is large (by hyperbolicity along \mathcal{F}) and contain respectively the endpoints and the center of $\mathcal{W}_{\varepsilon}^{cs}(y)$ which are separated by a uniform distance (of order ε).

We then build two disjoint curves J^-, J^+ through the endpoints of $\mathcal{W}_{\varepsilon}^{cu}(\varphi_T(y))$, tangent to $\mathcal{C}^{\mathcal{E}}$, disjoint from $\mathcal{W}_{\varepsilon}^{cs}(\varphi_T(y))$ and connecting $P_T(L^-)$ to $P_T(L^+)$. The curves $P_{-T}(J^-)$ and $P_{-T}(J^+)$ are still tangent to $\mathcal{C}^{\mathcal{E}}$, so that with L^-, L^+ they bound a box B with the required properties. The claim is thus proved. □

Let us choose r small. If $y, \varphi_T(y) \in H_x$ are r -close with $T \geq T_0$, the claim can be applied and moreover the projection of the boxes $\pi_x(B)$ and $\pi_x(P_T(B))$ intersect. If T_0 has been chosen large enough, using the uniform expansion along \mathcal{F} and the topological contraction along \mathcal{E} , one deduces that $B, P_T(B)$ are contained in small neighborhoods of $\mathcal{W}^{cs}(y)$ and $\mathcal{W}^{cu}(P_T(y))$ respectively. Consequently, the union of their projection on \mathcal{N}_x is diffeomorphic to $([-2, 2] \times [-1, 1]) \cup ([-1, 1] \times [-2, 2])$ in \mathbb{R}^2 : $\partial^{\mathcal{F}} B$ is identified with $\{-2, 2\} \times [-1, 1]$ and $\partial^{\mathcal{F}} P_T(B)$ is identified with $\{-1, 1\} \times [-2, 2]$.

The map $\tilde{P}_T := \pi_x P_T \pi_y$ is defined from $\pi_x(B)$ to $\pi_x(P_T(B))$. We can deform continuously the restriction $\tilde{P}_T: \partial(\pi_x(B)) \rightarrow \partial(\pi_x \circ P_T(B))$ so that it coincides after deformation with the restriction of a linear map $A: [-2, 2] \times [-1, 1] \rightarrow [-1, 1] \times [-2, 2]$ where $A = \begin{pmatrix} \pm 1/2 & 0 \\ 0 & \pm 2 \end{pmatrix}$, proving that the degree of the map

$$\Theta: z \mapsto \tilde{P}_T(z) - z / \|\tilde{P}_T(z) - z\|$$

from $\partial(\pi_x(B))$ to S^1 (for the canonical orientations of $\partial(\pi_x(B)) \subset \mathbb{R}^2$ and S^1) is non-zero. This proves that \tilde{P}_T has a fixed point p : otherwise, one can consider the degree of Θ on each circle $D_t := \partial([-2t, 2t] \times [-t, t])$ for $t \in (0, 1]$; it does not depend on t , hence is non-zero and it is a contradiction since for t small the disk bounded by D_t is disjoint from its image.

The point $\pi_y(p)$ belongs to B ; hence any iterate $P_s(\pi_y(p))$, $s \in [0, T]$, is contained $B(0, \delta) \subset \mathcal{N}_{\varphi_s(y)}$ by construction of B . \square

5.2.3 Construction of the box R

The box R is built from the next proposition. It implies the item 1 of Theorem 5.3.

Proposition 5.8. *For μ -almost every point $y \in H_x$, and any $\eta_x > 0$, there exist a generalized orbit $\bar{p} = (p(t))$ which is periodic and has two iterates \bar{p}_1, \bar{p}_2 satisfying:*

- \bar{p} is contained in the η_x -neighborhood of K ,
- \bar{p}_1, \bar{p}_2 are η_x -close to 0_y and (C_x, λ_x) -hyperbolic for \mathcal{F} .

The box $R \subset \mathcal{N}_x$ bounded by the four curves $\pi_x(\mathcal{W}^{cs}(\bar{p}_i))$ and $\pi_x(\mathcal{W}_{\alpha_x}^{cu}(\bar{p}_i))$, where $i \in \{1, 2\}$, has

- interior $\text{Int}(R)$ with positive μ_x -measure,
- boundary $\partial(R)$ with zero μ_x -measure.

Moreover if $z \in K$ is $r_0/2$ -close to x and $i \in \{1, 2\}$, then

- if $\pi_x(z) \in \pi_x(\mathcal{W}^{cs}(\bar{p}_i))$, the forward orbit of z accumulates on a periodic orbit of K .
- if $\pi_x(z) \in \pi_x(\mathcal{W}_{\alpha_x}^{cu}(\bar{p}_i))$, the backward orbit of z accumulates on a periodic orbit of K .

Since \bar{p}_1, \bar{p}_2 are arbitrarily close to $y \in H_x$, we have $R \subset B(0_x, \beta_x)$. A point $p(t)$ in the generalized orbit \bar{p} is a return of \bar{p} at x if its projection in K is $r_0/2$ -close to x .

Proof of Proposition 5.8. Recall the coherences of the center-stable plaques $\mathcal{W}^{cs}(\bar{u})$ defined for generalized orbits and the center-unstable plaques $\mathcal{W}_{\alpha_x}^{cu}(\bar{u})$ at (C_x, λ_x) -hyperbolic points for \mathcal{F} .

Lemma 5.9. *Consider any generalized orbit \bar{u} . If the projection $(y(t))$ of $(u(t))$ on K has arbitrarily large negative iterates in the r_0 -neighborhood of $K \setminus V$ and if $d(y(0), x) < r_0/2$, then, the projection $\pi_x(\mathcal{W}^{cs}(\bar{u}))$ has zero μ_x -measure.*

If \bar{u} is (C_x, λ_x) -hyperbolic for \mathcal{F} , then the same holds for the projection $\pi_x(\mathcal{W}_{\alpha_x}^{cu}(\bar{u}))$.

Proof. Assume by contradiction that $\pi_x(\mathcal{W}^{cs}(\bar{u}))$ has positive μ_x -measure: there exists a measurable set $A \subset K$ such that $\pi_x(A) \subset \pi_x(\mathcal{W}^{cs}(\bar{u}))$ and $\mu(A) > 0$. Hence there exist a positively recurrent point $z \in A$ and arbitrarily large $T > 0$ such that $\varphi_T(z)$ belongs to A , is arbitrarily close to z and $P_T(\mathcal{W}^{cs}(z))$ has arbitrarily small diameter (by topological hyperbolicity of \mathcal{E}). Since $y, z, \varphi_T(z)$ are $r_0/2$ -close to x , the coherence implies that $\pi_x(\mathcal{W}^{cs}(z)), \pi_x \circ P_T(\mathcal{W}^{cs}(z))$ and $\pi_x(\mathcal{W}^{cs}(u))$ are all contained in a same C^1 -curve. Hence, $\pi_x(\mathcal{W}^{cs}(z))$ contains $\pi_x \circ P_T(\mathcal{W}^{cs}(z))$. We have proved that $\mathcal{W}^{cs}(z)$ contains $\tilde{P}_T(\mathcal{W}^{cs}(z))$, where $\tilde{P}_T = \pi_z \circ P_T$, so that the sequence $(\tilde{P}_T^k(0_z))$ converges to a fixed point of \tilde{P}_T contained in $\mathcal{W}^{cs}(z)$. By Corollary 3.8, the orbit of z converges to a periodic orbit of K . This is a contradiction since μ -almost every point z in A has a forward orbit which is dense in the support of μ , which is not a periodic orbit by assumption.

The proof for center-unstable plaques is similar. If there exists a measurable set $A \subset K$ such that $\pi_x(A) \subset \pi_x(\mathcal{W}_{\alpha_x}^{cu}(\bar{u}))$ and $\mu(A) > 0$. Since the Lyapunov exponent of μ along \mathcal{F} is positive, up to reduce the set A , there exists $C' > 0, \lambda' > 1$ such that any point $z \in A$ is (C', λ') -hyperbolic for \mathcal{F} . By Proposition 3.38, there exists $\beta > 0$ such that $P_{-t}(\mathcal{W}_{\beta}^{cu}(z))$ is defined for any $t > 0$ and has a diameter smaller than $\alpha_{\mathcal{F}}\lambda^{-t/2}$. One ends the argument by considering z and $\varphi_{-T}(z)$ arbitrarily close to z . By the coherence in Lemma 5.5, the plaque $\mathcal{W}_{\alpha_x}^{cu}(\bar{u})$ is contained in $\mathcal{W}^{cu}(z)$ and in $\mathcal{W}^{cu}(\varphi_T(z))$. Hence if T is large enough, $P_{-T}(\mathcal{W}_{\beta}^{cu}(z))$ is arbitrarily small and contained in $\mathcal{W}_{\beta}^{cu}(z)$. We conclude as before. \square

Let us build a first approximation of R .

Lemma 5.10. *μ -almost every point $y \in H_x$ has arbitrarily large iterates $\varphi_t(y) \in H_x$ close to y such that the projection of the four plaques $\mathcal{W}^{cs}(y), \mathcal{W}^{cs}(\varphi_t(y)), \mathcal{W}_{\alpha_x}^{cu}(y), \mathcal{W}_{\alpha_x}^{cu}(\varphi_t(y))$ by π_x in \mathcal{N}_x bound a small box $R_y \subset B(0_x, \beta_x)$ whose measure for μ_x is positive.*

Proof. Let us choose $y \in H_x$ whose forward and backward orbits have dense sets of iterates in H_x and such that $\pi_x(\{z \in H_x, d(z, y) < r\})$ has positive μ_x -measure for any $r > 0$.

For $r > 0$ small, let us consider the four connected components of $B(\pi_x(y), r) \setminus \pi_x(\mathcal{W}^{cs}(y)) \cup \pi_x(\mathcal{W}_{\alpha_x}^{cu}(y))$. Since $\pi_x(\mathcal{W}^{cs}(y)), \pi_x(\mathcal{W}_{\alpha_x}^{cu}(y))$ have zero measure for μ_x , for one of these connected components Q , the measure $\mu_x(Q \cap H_x \cap B(\pi_x(y), r'))$ is positive for any $r' \in (0, r)$.

Choose $\varphi_t(y) \in H_x$ close to y in Q . By the coherence, the plaques $\mathcal{W}^{cs}(y), \mathcal{W}^{cs}(\varphi_t(y))$ have disjoint projection by π_x ; by Lemma 5.5, the same holds for the plaques $\mathcal{W}_{\alpha_x}^{cu}(y), \mathcal{W}_{\alpha_x}^{cu}(\varphi_t(y))$. Hence they bound a small box $R_y \subset B(0_x, \beta_x)$ whose measure for μ_x is positive. \square

End of the construction of the box R . By Lemma 5.10, for μ -almost every point $y \in H_x$, there exists $t > 0$ large such that the projection of the four plaques $\mathcal{W}^{cs}(y), \mathcal{W}^{cs}(\varphi_t(y)), \mathcal{W}_{\alpha_x}^{cu}(y), \mathcal{W}_{\alpha_x}^{cu}(\varphi_t(y))$ by π_x in \mathcal{N}_x bound a small box $R_y \subset B(0_x, \beta_x)$ with positive μ_x -measure.

We then choose $T > 0$ much larger than t such that $\varphi_T(y) \in H_x$ is very close to y and we apply Proposition 5.7. We get a point $p \in \mathcal{N}_x$ arbitrarily close to $\pi_x(y)$ and the repetition of the piece of orbit $\{p(s) = P_s(\pi_y(p)), s \in [0, T]\}$ gives a generalized periodic orbit \bar{p} which is η_x -close to K . Since y and $\varphi_t(y)$ are $(C_x/2, \lambda_x^2)$ -hyperbolic for \mathcal{F} , one deduces from Lemma 3.37 that \bar{p} and $\tilde{P}_t(\bar{p})$ are (C_x, λ_x) -hyperbolic for \mathcal{F} and have projection by π_x close to $\pi_x(y)$ and $\pi_x(\varphi_t(y))$.

The box $R \subset B(0_x, \beta_x)$ bounded by the projection of the four plaques $\mathcal{W}^{cs}(\bar{p}), \mathcal{W}^{cs}(\tilde{P}_t(\bar{p})), \mathcal{W}_{\alpha_x}^{cu}(\bar{p}), \mathcal{W}_{\alpha_x}^{cu}(\tilde{P}_t(\bar{p}))$ by π_x in \mathcal{N}_x is close to R_y , hence has positive μ_x -measure. By Lemma 5.9 the boundary of R has zero μ_x -measure.

Assume z is a point satisfying $\pi_x(z) \in \pi_x(\mathcal{W}^{cs}(\bar{p}_i))$. There exists $s \in [-1/4, 1/4]$ such that $z' := \varphi_s(z)$ is $r_0/2$ -close to y and still satisfies $\pi_y(z') \in \mathcal{W}^{cs}(\bar{p}_i)$. By the Global invariance, there exists $T' > 0$ such that the forward orbit of z' under $\pi_{z'} \circ P_{T'}$ is semi-conjugated by π_y with the forward orbit of $\pi_y(z')$ under $\pi_y \circ P_T$. The later converges to the fixed point $p_i := p_i(0)$ of the orbit $\bar{p}_i = (p_i(t))$. Corollary 3.8 applies and implies that the forward orbit of z' and z converges to a periodic orbit.

A similar argument holds when $\pi_x(z) \in \pi_x(\mathcal{W}_{\alpha_x}^{cu}(\bar{p}_i))$. This concludes Proposition 5.8. \square

Remark. We can choose the diameter of the rectangle R much smaller than β_x . In particular, by topological hyperbolicity of \mathcal{E} , if y is $r_0/2$ -close and $\pi_x(y) \in \text{Interior}(R)$, then any arc $I \subset \mathcal{W}^{cs}(y)$ satisfying $\pi_x(I) \subset R$ has forward iterates of length much smaller than β_x .

Moreover, for any $\beta_{box} > 0$, there exists a uniform time $t_1 > 0$ such that for any such y, I the length of $|P_t(I)|$ is much smaller than β_{box} when $t > t_1$.

5.2.4 New choices of constants

In the previous section we have built the box R . Before building the boxes B_1, \dots, B_k , we introduce $C_{\mathcal{F}}, \lambda_{\mathcal{F}}, \beta_{box}$ as in the statement of Theorem 5.3, and another constant $\alpha_{box} > 0$. One can reduce these numbers in order to satisfy the following properties:

- $C_{\mathcal{F}}, \lambda_{\mathcal{F}}$: by relaxing constants, one can require that for any $i \in \{1, 2\}$ and $s \in \mathbb{R}$, any generalized orbit \bar{u} satisfying $u(-t) \in \bar{P}_{-t-s}(\mathcal{W}_{\alpha_x}^{cu}(\bar{p}_i))$ for any $t \geq 0$ is $(C_{\mathcal{F}}, \lambda_{\mathcal{F}})$ -hyperbolic for \mathcal{F} .
- β_{box} : Proposition 3.40 associates to $C_{\mathcal{F}}, \lambda_{\mathcal{F}}$ some constants Δ, β . We can reduce β_{box} so that $\beta_{box} < \beta$.
- Δ_{box} : it is chosen so that the projection by the local diffeomorphisms π_x of any box with distortion Δ is a box with distortion Δ_{box} .
- α_{box} : one chooses α_{box} small so that the two following properties are satisfied (for the same reasons as in Section 5.2.1 for choosing α_x).

Backwards contraction. For any generalized orbit \bar{u} which is $(2C_{\mathcal{F}}, \lambda_{\mathcal{F}}^{1/2})$ -hyperbolic for \mathcal{F} , the set $\bar{P}_{-t}(\mathcal{W}_{\alpha_{box}}^{cu}(\bar{u}))$ is defined for any $t \geq 0$, is contained in $\mathcal{W}^{cu}(\bar{P}_{-t}(\bar{u}))$ and has diameter smaller than $\min(\beta_{box}, \alpha_{\mathcal{F}})\lambda_{\mathcal{F}}^{-t/2}$.

- *Coherence.* Consider two generalized orbits \bar{u}, \bar{v} and $t \geq 0$ such that:
 - $u(0) \in \mathcal{N}_y, v(-t) \in \mathcal{N}_{y'}$ with y, y' $r_0/2$ -close to x , and the projection $(y(t))_{t \in \mathbb{R}}$ of \bar{u} in K has arbitrarily large negative iterates in the r_0 -neighborhood of $K \setminus V$,
 - \bar{v} is $(2C_{\mathcal{F}}, \lambda_{\mathcal{F}}^{1/2})$ -hyperbolic for \mathcal{F} and $\pi_x(u(0)) \in \pi_x(\bar{P}_{-t}(\mathcal{W}_{\alpha_{box}}^{cu}(\bar{u})))$.

Then $\pi_x(\bar{P}_{-t}(\mathcal{W}_{\alpha_{box}}^{cu}(\bar{u}))) \subset \pi_x(\mathcal{W}^{cu}(\bar{v}))$.

5.2.5 Construction of the sub-boxes B_1, \dots, B_k

Proposition 5.11. *There exists $\delta > 0$ and finitely many sub-boxes $B_1, \dots, B_k \subset R$ with disjoint interiors, whose union contains $R \cap \pi_x(K)$, whose boundary has zero μ_x -measure, having the following properties.*

(i) *Geometry.* If γ_1, γ_2 are the two components of $\partial^{\mathcal{F}}(B_j)$, then

$$d(\gamma_1, \gamma_2) > 10 \cdot \max(\text{Diam}(\gamma_1), \text{Diam}(\gamma_2)).$$

(ii) *\mathcal{F} -boundary.* Any component γ of $\partial^{\mathcal{F}}(B_j)$ coincides with $\pi_x(I)$ of an arc I contained in $\pi_x(\bar{P}_{-t}(\mathcal{W}_{\alpha_x}^{cu}(\bar{p}_i)))$, $i \in \{1, 2\}$, where $\bar{P}_{-t}(\bar{p}_i)$ is a return of \bar{p}_i at x . Moreover, when it is defined, $\pi_x \circ \bar{P}_{-s}(I)$ for $s \geq 0$ is disjoint from all the $\text{Interior}(B_\ell)$, $\ell \in \{1, \dots, k\}$.

(iii) *\mathcal{E} -boundary.* Any component γ of $\partial^{\mathcal{E}}(B_j)$ satisfies one of the following properties:

- $\pi_x(\mathcal{W}^{cs}(y))$ does not intersect the δ -neighborhood of γ for any $y \in \pi_x(K) \cap B_j$; in particular $\gamma \cap \pi_x(K) = \emptyset$.
- γ is the projection by π_x of an arc I contained in the center-stable plaque $\mathcal{W}^{cs}(\bar{q})$ of a periodic generalized orbit \bar{q} . Moreover when it is defined, $\pi_x \circ \bar{P}_s(I)$ for $s \geq 0$ is disjoint from all the $\text{Interior}(B_\ell)$, $\ell \in \{1, \dots, k\}$.

(iv) Coherence with plaques. Consider y that is $r_0/2$ -close to x such that $\pi_x(y) \in B_i$. Then $\pi_x(\mathcal{W}^{cs}(y)) \cap B_i$ is an arc connecting the two components of $\partial^{\mathcal{F}} B_i$.

Consider y that is $r_0/2$ -close to x and a generalized orbit \bar{u} with $u(0) \in \mathcal{N}_y$, such that \bar{u} is $(2C_{\mathcal{F}}, \lambda_{\mathcal{F}}^{1/2})$ -hyperbolic for \mathcal{F} and $\pi_x(\bar{u}) \in B_i$. Then $\pi_x(\mathcal{W}_{\alpha_{\text{box}}}^{cu}(\bar{u})) \cap B_i$ is an arc connecting the two components of $\partial^{\mathcal{E}} B_i$.

This subsection is devoted to prove Proposition 5.11, which implies Item 2 of Theorem 5.3.

Unstable curves. Recall that R has been built from the plaques of \bar{p}_1, \bar{p}_2 in the orbit of \bar{p} . By the Local invariance, one chooses a finite set of iterates $\bar{p}_1 = \bar{p}$, $\bar{p}_2 = \bar{P}_{t_2}(\bar{p})$, $\bar{p}_3 = \bar{P}_{t_3}(\bar{p})$, \dots , of the generalized orbit \bar{p} such that:

- Each \bar{p}_i is a return of \bar{p} at x .
- Consider a return $\bar{P}_{-t}(\bar{p})$ of \bar{p} at x . Then there exists \bar{p}_j and $s \in [-1/4, 1/4]$ such that $\bar{P}_{-t}(\bar{p}) = \bar{P}_s(\bar{p}_j)$.
- If $\bar{P}_{-t}(\bar{p}_i) = \bar{p}_j$ for $|t| \leq 1/4$, then $i = j$.

One considers some C^1 -curves $\gamma_i^u \subset \mathcal{W}^{cu}(\bar{p}_i)$ and $\delta_1 > 0$ with the following properties:

- (a) The union of the curves $\pi_x(\gamma_i^u)$ is a C^1 -submanifold.
- (b) $\partial^{\mathcal{F}} R \subset \pi_x(\gamma_1^u) \cup \pi_x(\gamma_2^u)$.
- (c) Each γ_i^u locally coincides with $\bar{P}_{-s}(\mathcal{W}_{\alpha_x}^{cu}(\bar{p}_1))$ or $\bar{P}_{-s}(\mathcal{W}_{\alpha_x}^{cu}(\bar{p}_2))$ for some $s \geq 0$.
- (d) If $\bar{p}_i = \bar{P}_{-s}(\bar{p}_j)$ for some $s \geq 0$ and $i \neq j$ (so that $s \geq 1/4$), then $\bar{P}_{-s}(\gamma_j^u) \subset \gamma_i^u$.

Moreover $\pi_x(\gamma_i^u \setminus \bar{P}_{-s}(\gamma_j^u))$ is the union of two arcs of length larger than δ_1 .

- (e) For each i and each $t \geq 0$ such that $\bar{P}_{-t}(\bar{p}_i)$ is a return of \bar{p}_i at x , there exists j and $s \in [t - 1/4, t + 1/4]$ such that $\bar{p}_j = \bar{P}_{-s}(\bar{p}_i)$ and $\pi_x \circ \bar{P}_{-s}(\gamma_i^u) = \pi_x \circ \bar{P}_{-t}(\gamma_i^u)$.

Property (d) shows that if one chooses sub-curves $\tilde{\gamma}_i^u \subset \gamma_i^u$ such that the components of $\pi_x(\gamma_i^u \setminus \tilde{\gamma}_i^u)$ have length smaller than δ_1 , then the inclusion $\bar{P}_{-s}(\tilde{\gamma}_j^u) \subset \tilde{\gamma}_i^u$ still holds when $\bar{p}_i = \bar{P}_{-s}(\bar{p}_j)$ for some $s \geq 0$ and $i \neq j$.

Let us explain how to build it. Let $\sigma_1 \subset \mathcal{W}_{\alpha_x}^{cu}(\bar{p}_1)$ such that $\pi_x(\sigma_1)$ is a component of $\partial^{\mathcal{F}} R$. The returns of the backward iterates of \bar{p}_1 by \bar{P}_t inside the finite set $\{\bar{p}_1, \bar{p}_2, \dots\}$ defines an infinite periodic sequence $\bar{z}_1, \bar{z}_2, \dots$. There exists a minimal $s_k > 0$ such that $\bar{P}_{-s_k}(\bar{z}_k) = \bar{z}_{k+1}$.

We inductively define σ_k as the curve in $\mathcal{W}^{cu}(\bar{z}_k)$ such that $\pi_x(\sigma_k)$ is the δ_1 -neighborhood of $\pi_x(\bar{P}_{-s_k}(\sigma_{k-1}))$. We then define γ_i^1 as the union of the σ_k such that $\bar{z}_k = \bar{p}_i$. Since $\bar{P}_{-s}(\mathcal{W}_{\alpha_x}^{cu}(\bar{p}_1))$ decreases exponentially as $s \rightarrow +\infty$, by choosing δ_1 small enough, we get:

- For any i , there is $s \geq 0$ such that $\gamma_i^1 \subset \bar{P}_{-s}(\mathcal{W}_{\alpha_x}^{cu}(\bar{p}_1))$.
- If $\bar{p}_i = \bar{P}_{-s}(\bar{p}_j)$ for some $s \geq 0$ and $i \neq j$, then $\bar{P}_{-s}(\gamma_j^1) \subset \gamma_i^1$ and $\pi_x(\gamma_i^1 \setminus \bar{P}_{-s}(\gamma_j^1))$ is the union of two arcs of length larger than δ_1 .

We repeat the same construction starting with the curve in $\mathcal{W}_{\alpha_x}^{cu}(\bar{p}_2)$ which projects to the other component of $\partial^{\mathcal{F}}R$. One obtains another family of curves γ_i^2 . One then set $\gamma_i^u = \gamma_i^1 \cup \gamma_i^2$ so that items (b), (c), (d) are satisfied.

In order to check item (a), it is enough to notice that (from item (c) and Lemma 5.5, the union of any two curves $\pi_x(\gamma_i^u) \cup \pi_x(\gamma_j^u)$ is a C^1 -submanifold.

Let us prove item (e): if $\bar{P}_{-t}(\bar{p}_i)$ is a return of \bar{p}_i at x , by definition of the family $\bar{p}_1, \bar{p}_2, \bar{p}_3, \dots$, there exists j and $s \in [t - 1/4, t + 1/4]$ such that $\bar{P}_{-s}(\bar{p}_i) = \bar{p}_j$. By the Local invariance, $\pi_x \circ P_{-t}(\gamma_i^u) = \pi_x \circ \bar{P}_{-s}(\gamma_i^u)$.

Stable curves. Let us consider $\delta_2 > 0$ much smaller than $\min(\delta_1, \alpha_{box})$. One also considers $\varepsilon, \delta_3 > 0$ that will be fixed later.

Choose a point y in a subset of full μ -measure of $H_x \cap \pi_x^{-1}(\text{Interior}(R))$ and a forward iterate $\varphi_T(y) \in H_x$ close to y with $T > 0$ large. By Proposition 5.7, we build a periodic generalized orbit \bar{q} which is ε -close to the zero-section of \mathcal{N} for the Hausdorff distance. As for the generalized orbit \bar{p} , one chooses a finite set of iterates $\bar{q}_1, \bar{q}_2, \bar{q}_3, \dots$, of the generalized orbit \bar{q} such that:

- Each \bar{q}_i is a return of \bar{q} at x .
- Assume that $\bar{P}_{-t}(\bar{q})$ is a return of \bar{q} at x . Then there exists \bar{q}_j and $s \in [-1/4, 1/4]$ such that $\bar{P}_{-t}(\bar{q}) = \bar{P}_s(\bar{q}_j)$.
- If $\bar{P}_{-t}(\bar{q}_i) = \bar{q}_j$ for $|t| \leq 1/4$, then $i = j$.

We denote by $\{z_1, \dots, z_m\}$ the collection of \bar{p}_i, \bar{q}_j and build curves $\gamma_1^s, \dots, \gamma_m^s$ such that:

- (a') The union of the curves $\pi_x(\gamma_i^s)$ is a C^1 -submanifold.
- (b') $\partial^{\mathcal{E}}R \subset \pi_x(\gamma_1^s) \cup \pi_x(\gamma_2^s)$.
- (c') Each γ_i^s is contained in $\mathcal{W}^{cs}(z_i)$.
- (d') If $z_i = \bar{P}_s(z_j)$ for some $s \geq 0$ and $i \neq j$, then $\bar{P}_s(\gamma_j^s) \subset \gamma_i^s$.
- (e') For each i and each $t \geq 0$ such that $\bar{P}_{-t}(z_i)$ is a return of z_i at x , there exists j and $s \in [t - 1/4, t + 1/4]$ such that $z_j = \bar{P}_{-s}(z_i)$ and $\pi_x \circ \bar{P}_{-s}(\gamma_i^s) = \pi_x \circ \bar{P}_{-t}(\gamma_i^s)$.
- (f') If $\gamma_i^s \cap R$ is non-empty, then it is an arc which connects the two components of $\partial^{\mathcal{F}}R$.

The center-stable plaques $\mathcal{W}^{cs}(z_i)$ satisfy the properties of items (a') to (e') above, by coherence, trapping, the Local invariance and definition of R . Note that these properties are still satisfied if one replace $\mathcal{W}^{cs}(z_i)$ by a sub-arc γ_i^s such that the components of $\pi_x(\mathcal{W}^{cs}(z_i) \setminus \gamma_i^s)$ have length smaller than β_x . (For the property (d') this comes from the choice of β_x and the fact that $z_i = \bar{P}_s(z_j)$ for some $s \geq 0$ and $i \neq j$ implies $s \geq 1/4$.) By construction, R is contained in $B(0, \beta_x)$, so one can find such sub-arcs γ_i^s which satisfy (f').

Strips. The set $\text{Interior}(R) \setminus (\pi_x(\gamma_1^s) \cup \dots \cup \pi_x(\gamma_m^s))$ has finitely many connected components, whose closures are center-stable sub-boxes of R that we call strips.

We distinguish two kinds of strips:

- *thin strips*: strips whose \mathcal{E} -boundaries are δ_2 -close to each-other: any C^1 -curve in the strip which connects the two components of the \mathcal{E} -boundary and which is tangent to $\mathcal{C}^{\mathcal{F}}$ has length smaller than δ_2 ,
- *thick strips*: the other ones.

Lemma 5.12. *The minimal distance between the components of the \mathcal{E} -boundary of the thick strips is bounded away from 0 (uniformly in the choice of the periodic orbit \bar{q}).*

Proof. Otherwise, there exist z_i, z_j and γ_i^s, γ_j^s whose projections by π_x have two points arbitrarily close, and there exists a transverse arc tangent to \mathcal{C}^F of length larger than δ_2 connecting these two curves.

Taking the limit, one gets two points \bar{z}, \bar{z}' which still belong to generalized orbits (η -close to the 0-section) whose center-stable plaques intersect but are not contained in a same C^1 -submanifold tangent to \mathcal{C}^E . This contradicts the coherence. \square

The next lemma fixes the constant $\varepsilon > 0$.

Lemma 5.13. *Let us choose $\delta_3 > 0$. If ε is small enough, then for any thick strip S and any $y \in K$ which is $r_0/2$ -close to x and satisfies $\pi_x(y) \in S$, $\pi_x(\mathcal{W}^{cs}(y)) \cap S$ is δ_3 -close to $\partial^E S$.*

Proof. First we show that when ε goes to zero, the distance between $\pi_x(K) \cap S$ and $\partial^E S$ goes to zero. Otherwise there exists a thick strip S and a point $y \in K$ that is $r_0/2$ -close to x such that $\pi_x(y)$ belongs to S and is at a distance from $\partial^E S$, larger than a uniform constant $e > 0$. For $\varepsilon > 0$ small enough, there exists a point z_i close to y , defining a curve γ_i^s which intersects S but is at a bounded distance from $\partial^E S$. This contradicts the definition of the strips S as the closures of connected components of $\text{Interior}(R) \setminus (\pi_x(\gamma_1^s) \cup \dots \cup \pi_x(\gamma_m^s))$.

We then conclude that all the set $\pi_x(\mathcal{W}^{cs}(y)) \cap S$ is close to $\partial^E S$. Otherwise, one can take a limit \bar{y} of such points y and a limit γ of components of $\partial^E S$, such that $\pi_x(\mathcal{W}^{cs}(\bar{y}))$ and γ intersect at $\pi_x(\bar{y})$ but are not contained in a C^1 -curve. Since γ is contained in a center-stable plaque, this contradicts the coherence. \square

By Lemmas 5.12 and 5.13, we take $\delta_3 \in (0, \delta_2)$ smaller than half of the minimal distance between the components of $\partial^E S$ of any thick strips S and choose ε such that in each thick strip S , the sets $\pi_x(\mathcal{W}^{cs}(y)) \cap S$, for any $y \in S$, is at a distance to $\partial^E S$ smaller than δ_3 . This allows to build in each thick strip S , two disjoint center-stable sub-boxes (sub-strips) S_-, S_+ such that:

- $S_- \cup S_+$ contains $\pi_x(K) \cap S$,
- the two components of $\partial^E S_-$ (resp. $\partial^E S_+$) are δ_2 -close,
- one component of $\partial^E S_-$ (resp. $\partial^E S_+$) coincides with a component of $\partial^E S$, the other one is disjoint from the δ_3 -neighborhood of $\partial^E S$.

In particular, there exists $\delta > 0$ such that for any thick strip S and any $y \in \pi_x(K) \cap S$, the plaque $\pi_x(\mathcal{W}^{cs}(y))$ is disjoint from the δ -neighborhood of the component γ of $\partial^E S_{\pm}$ which is not contained in $\partial^E S$.

The sub-boxes B_0, \dots, B_k . One chooses sub-curves $\tilde{\gamma}_i^u \subset \gamma_i^u$ such that the components of $\pi_x(\gamma_i^u \setminus \tilde{\gamma}_i^u)$ have length smaller than δ_1 and whose endpoints do not belong to the interior of thin strips nor to boxes S_{\pm} associated to a thick strip S .

The sub-boxes B_0, \dots, B_k are obtained from a thin strips $S_0 = S$ or a sub-strips $S_0 \in \{S_-, S_+\}$ as follow: We consider the connected components of

$$\text{Interior}(S_0) \setminus (\pi_x(\tilde{\gamma}_1^u) \cup \dots \cup \pi_x(\tilde{\gamma}_m^u))$$

and take their closures. They have disjoint interiors and their union contains $R \cap \pi_x(K)$. The item (i) holds by the choice of δ_2 , much smaller than the distance between pair of curves γ_i^u .

Each component of the \mathcal{F} -boundary of these boxes is the projection $\pi_x(I)$ of an arc I contained in a curve $\tilde{\gamma}_i^u$. In particular the \mathcal{F} -boundary has zero μ_x -measure. Moreover, by the properties on the curves $\tilde{\gamma}_i^u$ and the Local invariance, for each return $\bar{P}_{-s}(\bar{p}_i)$ at x , $s \geq 0$, the iterate $\pi_x(\bar{P}_{-s}(I))$ is contained in a curve $\tilde{\gamma}_j^u$, hence is disjoint from the interior of the boxes B_{ℓ} . The item (ii) is thus satisfied.

For each component γ of the \mathcal{E} -boundary of these boxes B_ℓ , either the δ -neighborhood of γ is disjoint from $\pi_x(\mathcal{W}^{cs}(y))$ for any $y \in B_\ell$, or γ is the projection $\pi_x(I)$ of an arc I contained in a curve $\tilde{\gamma}_i^s$. In particular the \mathcal{E} -boundary has zero μ_x -measure. Moreover, in this second case by the properties on the curves $\tilde{\gamma}_i^s$ and Local invariance, for each return $\bar{P}_{-s}(\bar{z}_i)$ at x , $s \geq 0$, the iterate $\pi_x(\bar{P}_{-s}(I))$ is contained in a curve $\tilde{\gamma}_j^s$, hence is disjoint from the interior of the boxes B_ℓ . The item (iii) is thus satisfied.

We have proved that the boundary of the boxes has zero μ_x -measure.

Coherence with plaques. Consider $y \in K$ that is $r_0/2$ -close to x and such that $\pi_x(y) \in B_i$. Let γ be a component of $\partial^\mathcal{E} B_i$. There are two cases.

- If γ is contained in a curve γ_i^s (hence in a center-stable plaque), then by coherence, $\pi_x(\mathcal{W}^{cs}(y))$ is disjoint from or contains γ .
- Otherwise B_i is built from a sub-box S_- or S_+ of a thick strip S , and γ is disjoint from $\pi_x(\mathcal{W}^{cs}(y'))$ for any $y' \in \pi_x(K) \cap B_i$. So $\pi_x(\mathcal{W}^{cs}(y)) \cap B_i$ is disjoint from γ .

We have obtained the first part of item (iv).

In order to check the second part, one recalls that the components of the \mathcal{F} -boundary of each box B_i are in a curve $\pi_x(\bar{P}_{-s}(\mathcal{W}_{\alpha_x}^{cu}(\bar{p}_i)))$, where \bar{p}_i is (C_x, λ_x) -hyperbolic for \mathcal{F} by item (c) above; the length of all their backwards iterates of $\mathcal{W}_{\alpha_x}^{cu}(\bar{p}_i)$ remain small. If y is $r_0/2$ -close to x and \bar{u} is a generalized orbit with $u(0) \in \mathcal{N}_y$, such that $\pi_x(\bar{u}) \in B_i$ and \bar{u} is $(2C_{\mathcal{F}}, \lambda_{\mathcal{F}}^{1/2})$ -hyperbolic for \mathcal{F} , then the lengths of all their backwards iterates of $\mathcal{W}_{\alpha_{box}}^{cu}(\bar{u})$ remain small. Hence Lemma 5.5 implies that the union of $\pi_x(\mathcal{W}^{cu}(\bar{u}))$ and of $\partial^\mathcal{F} B_i$ is a submanifold. Consequently, each component of $\partial^\mathcal{F} B_i$ is either disjoint from or contained in $\pi_x(\mathcal{W}^{cu}(\bar{u}))$. Since the distance between the two components of $\partial^\mathcal{E} B_i$ is much smaller than α_{box} , the curve $\pi_x(\mathcal{W}_{\alpha_{box}}^{cu}(\bar{u}))$ meets both of them. This gives the second part of item (iv) and thus completes the proof of Proposition 5.11. \square

5.3 The Markovian property

We have proved the items 1 and 2 of Theorem 5.3 in Sections 5.2.3 and 5.2.5. We now prove the other items.

Items 3 and 4 of Theorem 5.3. Let us consider a transition (y, t) between two sub-boxes $B, B' \in \{B_1, \dots, B_k\}$ (associated to the constants $C_{\mathcal{F}}, \lambda_{\mathcal{F}}$) and such that $t > t_{box}$, where t_{box} is a large constant to be chosen later. By item (iv) of Proposition 5.11, there exists an arc $I \subset \mathcal{W}^{cs}(y)$ containing 0_y , whose projection by π_x is contained in B and connects the two components of the \mathcal{F} -boundary of B .

Lemma 5.14. *The image $\pi_x(P_t(I))$ is contained in B' .*

Proof. Otherwise, by item (iv) of Proposition 5.11, there exists $u' \in P_t(I)$ which is not an endpoint and which projects by π_x inside the \mathcal{F} -boundary of B' . Hence by the item (iii) of Proposition 5.11 and the Global invariance, $u := P_{-t}(u')$ projects by π_x is contained in $\partial^\mathcal{F}(B)$. Since u is not an endpoint of I , the arc $\pi_x(I)$ is not contained in B . This is a contradiction. \square

Lemma 5.15. *There exists T_1 such that provided $t > T_1$, each endpoint of I belongs to a generalized orbit \bar{u} such that $\bar{P}_t(\bar{u})$ is $(2C_{\mathcal{F}}, \lambda_{\mathcal{F}}^{1/2})$ -hyperbolic for \mathcal{F} .*

Proof. Let u be an endpoint of I . Note that $u \in \mathcal{W}_{\beta_x}^{cs}(y)$. By item (ii) of Proposition 5.11, $\pi_x(u)$ belongs to $\pi_x(\bar{P}_{-\tau}(\mathcal{W}_{\alpha_x}^{cu}(\bar{p}_i)))$, for some return $\bar{P}_{-\tau}(\bar{p}_i)$, $\tau > 0$ of \bar{p}_i , $i = 1, 2$ at x . By definition of the \bar{p}_i , one can assume that there is no discontinuity in the orbit $\bar{p}_i(-\tau - s)$, with $s \in [-1, 0]$. Lemma 5.6 and Remark 5.2.1 apply and define the generalized orbit \bar{u} . It is $(C_{\mathcal{F}}, \lambda_{\mathcal{F}})$ -hyperbolic for \mathcal{F} by our choice of $C_{\mathcal{F}}$ and $\lambda_{\mathcal{F}}$.

For any $\varepsilon > 0$ small, there exist uniform $T_0, C_0 > 0$ such that the following holds.

- Since \mathcal{E} is topologically contracted, the points $P_s(y)$ and $u(s)$ are close for any $s > T_0$. Moreover, $(y, \varphi_t(y))$ is $(C_{\mathcal{F}}, \lambda_{\mathcal{F}})$ -hyperbolic for \mathcal{F} . Hence, for $s \in (T_0, t)$,

$$\|DP_{s-t}|_{\mathcal{F}}(u(t))\| \leq \|DP_{s-t}|_{\mathcal{F}}(\varphi_t(y))\|(1+\varepsilon)^{(t-s)} \leq C_{\mathcal{F}}\lambda_{\mathcal{F}}^{s-t}(1+\varepsilon)^{(t-s)},$$

- $\|DP_{T_0-s}|_{\mathcal{F}}(u(T_0))\| \leq C_0$ for any $s \in [0, T_0]$.

So $\bar{P}_t(\bar{u})$ is $(2C_{\mathcal{F}}, \lambda_{\mathcal{F}}^{1/2})$ -hyperbolic for \mathcal{F} if T_1 satisfies

$$C_0\lambda_{\mathcal{F}}^{T_0}(1+\varepsilon)^{-T_0} < (1+\varepsilon)^{-T_1}\lambda_{\mathcal{F}}^{T_1/2}.$$

□

Let \bar{u} and \bar{v} be the generalized orbits associated to each endpoint of I . By coherence (stated in Section 5.2.4), the projection of the plaques $\mathcal{W}_{\alpha_{box}}^{cu}(\bar{P}_t(\bar{u}))$, $\mathcal{W}_{\alpha_{box}}^{cu}(\bar{P}_t(\bar{v}))$ are disjoint and by item (iv) of Proposition 5.11 they cross B' . We thus obtain a center-unstable sub-box $B^{cu} \subset B'$ bounded by these curves.

For all $0 \leq s \leq t$, the iterates $P_{-s} \circ \pi_{\varphi_t(y)}(B^{cu})$ are contained in $B(0, 2\alpha_0) \subset \mathcal{N}_{\varphi_{t-s}(y)}$, where α_0 is an upper bound on the size of the plaques \mathcal{W}^{cs} and \mathcal{W}^{cu} . Moreover, the four edges remain tangent to the cones $\mathcal{C}^{\mathcal{E}}$ or $\mathcal{C}^{\mathcal{F}}$. We denote $B^{cs} := \pi_x \circ P_{-t} \circ \pi_{\varphi_t(y)}(B^{cu})$.

Lemma 5.16. *The sets $P_s(\pi_y(B^{cs}))$ have diameter smaller than β_x for each $s \in [0, t]$. Their diameter is smaller than β_{box} when s is larger than some uniform constant T_2 .*

Proof. There exists a uniform $C > 0$ such that the diameter of $P_s(\pi_y(B^{cs})) = P_{s-t}(\pi_{\varphi_t(y)}(B^{cu}))$ is smaller than $C \max(|P_s(I)|, |P_{s-t}(\partial^{\mathcal{F}} B^{cu})|)$.

By our choice of α_{box} , the curve $P_{s-t}(\mathcal{W}_{\alpha_{box}}^{cu}(u(t)))$ has a size much smaller than β_{box} and it contains $P_{s-t}(\partial^{\mathcal{F}} B^{cu})$. By Remark 5.2.3, the length of $P_s(I)$ is much smaller than β_x , implying the first property, since \mathcal{E} is topologically contracted. This length is much smaller than β_{box} when t is larger than a constant T_2 , giving the second property. □

The following lemmas end the proof of item 3 of Theorem 5.3.

Lemma 5.17. *When t is larger than some T_3 , the δ -neighborhood of $\pi_x(I)$ contains B^{cs} .*

Proof. For each point z in B^{cs} , there exists a curve $\gamma \subset \pi_y(B^{cs})$ tangent to $\mathcal{C}^{\mathcal{F}}$ which connects $\pi_y(z)$ to a point z' in I . The iterates $P_s(\gamma)$ for $s \in [0, t]$ are still tangent to $\mathcal{C}^{\mathcal{F}}$ and, for any such s which is larger than some uniform T , one has:

- $P_s(\gamma)$ is contained in $P_s(\pi_y(B^{cs}))$ hence in a small neighborhood of $0_{\varphi_s(y)}$ by Lemma 5.16,
- the tangent spaces to $P_s(\gamma)$ are close to $\mathcal{F}(\varphi_s(y))$.

The derivative of P_{t-s} along $P_s(\gamma)$ and $\mathcal{F}(\varphi_s(y))$ can be compared. Since $(y, \varphi_t(y))$ is $(C_{\mathcal{F}}, \lambda_{\mathcal{F}})$ -hyperbolic for \mathcal{F} , $|P_s(\gamma)|$ is exponentially small in $t-s$ for $s \geq T$. If t is large enough, this implies that the length of γ is exponentially small in t . In particular it is smaller than δ . □

Lemma 5.18. *When $t > T_3$, the box B^{cs} is a center-stable sub-box of B .*

Proof. By coherence (stated in Section 5.2.4), the union of the \mathcal{F} -boundary of B and B^{cs} is contained in the union of two disjoint C^1 -curves. If one assumes by contradiction that B^{cs} is not contained in B , there exists a point u in the \mathcal{E} -boundary of B that belongs to $\text{Interior}(B^{cs})$. By item (iii) of Proposition 5.11, two cases are possible:

- u belongs to the projection by π_x of an arc $J \subset \mathcal{W}^{cs}(\bar{q})$ of a generalized periodic orbit \bar{q} and no forward iterate of J projects to the interior of any box B_1, \dots, B_k . This is a contradiction since $\pi_x \circ P_t \circ \pi_y(u)$ belongs to the interior of $B^{cu} \subset B'$.
- u is δ -far from $\pi_x(\mathcal{W}^{cs}(y))$: it contradicts the previous lemma.

We have proved that $B^{cs} \subset B$ and $\partial^{\mathcal{F}} B^{cs} \subset \partial^{\mathcal{F}} B$ as required. \square

The following ends the proof of the item 4 of Theorem 5.3.

Lemma 5.19. *The distortion of B^{cu} is bounded by Δ_{box} .*

Proof. Since B^{cs} is a center-stable sub-box of B and since B satisfies the item (i) of Proposition 5.11, the box B^{cs} satisfies this condition too. Since $(y, \varphi_t(y))$ is $(C_{\mathcal{F}}, \lambda_{\mathcal{F}})$ -hyperbolic for \mathcal{F} and since the diameter of $\pi_y(B^{cu})$ is smaller than β_{box} , Proposition 3.40 and Remark 3.41 (and the choice of the constants Δ, β_{box}) imply that $\pi_y(B^{cu})$ has distortion bounded by Δ . From our choice of Δ_{box} , the box B^{cu} has distortion bounded by Δ_{box} . \square

Lemma 5.20. *Consider $0 < s < t$ such that s and $t - s$ are larger than some T_4 and such that $\varphi_s(y)$ is $r_0/2$ -close to x and $\pi_x(\varphi_t(y))$ belongs to some box B_i . Then the interior of $\pi_x \circ P_s \circ \pi_y(B^{cs})$ does not meet the \mathcal{E} -boundaries of B_i .*

Proof. Note that if T_4 is large enough, then the diameter of $\pi_x \circ P_s \circ \pi_y(B^{cs})$ is arbitrarily small: indeed, from the topological contraction of \mathcal{E} , the distance between the two components of the \mathcal{F} -boundary of $P_s \circ \pi_y(B^{cs})$ is arbitrarily small if s is large enough. These components are contained in the backward image of plaques $\mathcal{W}_{\alpha_{box}}^{cu}(\bar{P}_t(\bar{u}))$, $\mathcal{W}_{\alpha_{box}}^{cu}(\bar{P}_t(\bar{v}))$ whose lengths are exponentially small in $t - s$.

Assume by contradiction that the interior of $\pi_x \circ P_s \circ \pi_y(B^{cs})$ meets some component γ of $\partial^{\mathcal{E}} B_i$. By item (iii) of Proposition 5.11, γ satisfies one of the two next cases.

- γ is disjoint from the δ -neighborhood of $\pi_x(K) \cap B_i$: it is a contradiction since $\pi_x \circ P_s \circ \pi_y(B^{cs})$ contains $\pi_x(\varphi_s(y)) \in \pi_x(K) \cap B_i$ and has arbitrarily small diameter.
- γ is the projection by π_x of an arc I contained in the center-stable plaque $\mathcal{W}^{cs}(\bar{q})$ of a periodic generalized orbit \bar{q} and $\pi_x \circ \bar{P}_\tau(I)$ for $\tau \geq 0$ is disjoint from all the $\text{Interior}(B_\ell)$, $\ell \in \{1, \dots, k\}$. This is a contradiction since by the Global invariance, there exists an iterate $\pi_x \circ \bar{P}_\tau(I)$ which intersects the interior of B^{cu} .

\square

We take t_{box} equal to the supremum of the T_i for $i = 1, 2, 3, 4$.

Proof of items 5 and 6 of Theorem 5.3. We consider two transitions $(y_1, t_1), (y_2, t_2)$ with $t_1, t_2 > t_{box}$ such that the interiors of the boxes B_1^{cu} and B_2^{cu} intersect.

Lemma 5.21. *Assume that there exists $\theta \in \text{Lip}_2$ such that:*

- $\theta(t_1) \geq t_2 - 2$, $\theta^{-1}(t_2) \geq t_1 - 2$ and $\theta(0) \geq -1$,
- $d(\varphi_t(y_1), \varphi_{\theta(t)}(y_2)) < r_0/2$ for $t \in [0, t_1] \cap \theta^{-1}([0, t_2])$.

Then $B_2^{cu} \subset B_1^{cu}$.

Proof. Let us assume by contradiction that $\partial^{\mathcal{F}}(B_1^{cu}) \cap \text{Interior}(B_2^{cu}) \neq \emptyset$. The two transitions are associated to boxes B_1, B_2 (containing y_1, y_2 respectively) and B' containing both B_1^{cu} and B_2^{cu} . We also denote $y'_1 := \varphi_{t_1}(y_1)$ and $y'_2 := \varphi_{t_2}(y_2)$. Moreover we set $[a, b] = [0, t_1] \cap \theta^{-1}([0, t_2])$.

By the Global invariance

$$\pi_x \circ P_{a-b} \circ \pi_{\varphi_b(y_1)}(B_i^{cu}) = \pi_x \circ P_{\theta(a)-\theta(b)} \circ \pi_{\varphi_{\theta(b)}(y_2)}(B_i^{cu}).$$

By our assumptions, $|\theta(b) - t_2| \leq 2$ and $|b - t_1| \leq 2$. By the Local invariance,

$$\pi_{\varphi_b(y_1)}(B_i^{cu}) = P_{b-t_1} \circ \pi_{y'_1}(B_i^{cu}) \text{ and } \pi_{\varphi_{\theta(b)}(y_2)}(B_i^{cu}) = P_{\theta(b)-t_2} \circ \pi_{y'_2}(B_i^{cu}).$$

Since θ is 2-Lipschitz and $\theta(0) \geq -1$, we check that $|a| \leq 2$ and that $\varphi_a(y_1)$ is r_0 -close to x . Hence by the Local invariance,

$$\pi_x \circ P_{a-b} \circ \pi_{\varphi_b(y_1)}(B_i^{cu}) = \pi_x \circ P_{-b} \circ \pi_{\varphi_b(y_1)}(B_i^{cu}).$$

This shows that

$$\pi_x \circ P_{a-b} \circ \pi_{\varphi_b(y_1)}(B_1^{cu}) = B_1^{cs} \text{ and } \pi_x \circ P_{\theta(a)-\theta(b)} \circ \pi_{\varphi_{\theta(b)}(y_2)}(B_2^{cu}) = \pi_x \circ P_{\theta(a)} \circ \pi_{y_2}(B_2^{cs}).$$

Consequently, the interior of $\pi_x \circ P_{\theta(a)} \circ \pi_{y_2}(B_2^{cs})$ meets the \mathcal{F} -boundary of B_1^{cs} . We denote by γ the corresponding component of $\partial^{\mathcal{F}} B_1^{cs}$. By item (ii) of Proposition 5.11 we have $\gamma = \pi_x(I)$ where I is an arc in $\bar{P}_{-t}(\mathcal{W}_{\alpha_x}^{cu}(\bar{p}_1))$ or in $\bar{P}_{-t}(\mathcal{W}_{\alpha_x}^{cu}(\bar{p}_2))$ for some $t \geq 0$.

In the case $\theta(a) \in [0, 1]$, the Local invariance shows that $\pi_x \circ P_{\theta(a)} \circ \pi_{y_2}(B_2^{cs}) = B_2^{cs}$. Hence the interior of B_2^{cs} meets $\pi_x(I)$, contradicting the item (ii) of Proposition 5.11.

In the other case, $\theta(a) > 1$. The Global invariance (Remark 3.5, item (e)) shows that since $\pi_x(I)$ intersects $\text{Interior}(\pi_x \circ P_{\theta(a)} \circ \pi_{y_2}(B_2^{cs}))$, there is an iterate $\bar{P}_{-s}(I)$, $s \geq 0$, whose projection by π_x meets $\text{Interior}(B_2^{cs})$. Again, this contradicts the item (ii) of Proposition 5.11. \square

In order to prove item 5, we have to check that, up to exchange (y_1, t_1) and (y_2, t_2) , the conditions of Lemma 5.21 are satisfied by some θ which furthermore satisfies $|\theta(t_1) - t_2| \leq 1/2$. By the Global invariance (Remark 3.5, item (e)), there exists $\theta \in \text{Lip}_2$ such that:

- $\theta(t_1) \in [t_2 - 1/4, t_2 + 1/4]$,
- $d(\varphi_t(y_1), \varphi_{\theta(t)}(y_2)) < r_0/2$ for $t \in [0, t_1] \cap \theta^{-1}([0, t_2])$.

Note that it also gives $|\theta^{-1}(t_2) - t_1| \leq 1/2$ since θ is 2-bi-Lipschitz. Up to exchange y_1 and y_2 , we can suppose $0 \in \theta^{-1}([0, t_2])$, hence the assumptions of the Lemma 5.21 are satisfied. This gives $B_2^{cu} \subset B_1^{cu}$ and ends the proof of item 5.

The proof of item 6 uses similar ideas. Let B_i^{cs}, B_i^{cu} , $i = 1, 2$, be the boxes associated to the transitions (y, t_1) and (y, t_2) where $y = y_1 = y_2$ such that the interior of B_1^{cs} and B_2^{cs} intersect and that $t_2 > t_1 + t_{box} > 2t_{box}$. Recall that y belongs to the interior of some box $B \in \{B_1, \dots, B_k\}$ which contains B_1^{cs} and B_2^{cs} . Let I be the arc in $\mathcal{W}^{cs}(y)$ connecting the two components of $\partial^{\mathcal{F}} B$ and let \bar{u}, \bar{v} be the two generalized orbits associated to the endpoints of I as in the proof of items 3 and 4 above.

Let us consider the two boxes $B_1^{cu} = \pi_x P_{t_1} \pi_y(B_1^{cs})$ and $\pi_x P_{t_1} \pi_y(B_2^{cs})$: their interior intersect (since they contain $\pi_x \varphi_{t_1}(y)$). By construction $\partial^{\mathcal{F}} B_1^{cu}$ is contained in the union of $\pi_x(\mathcal{W}^{cu}(\bar{P}_{t_1}(\bar{u})))$ and $\pi_x(\mathcal{W}^{cu}(\bar{P}_{t_1}(\bar{v})))$. By construction, the \mathcal{F} -boundary of $P_{t_2} \pi_y(B_2^{cs})$ is contained in the union of $\mathcal{W}_{\alpha_{box}}^{cu}(\bar{P}_{t_2}(\bar{u}))$ and $\mathcal{W}_{\alpha_{box}}^{cu}(\bar{P}_{t_2}(\bar{v}))$ and by the coherence stated in Section 5.2.4 the \mathcal{F} -boundary of $\pi_x P_{t_1} \pi_y(B_2^{cs})$ is also contained in the union of the projection by π_x of the plaques $\mathcal{W}^{cu}(\bar{P}_{t_1}(\bar{u}))$ and $\mathcal{W}^{cu}(\bar{P}_{t_1}(\bar{v}))$. Moreover by applying Lemma 5.20 to $\varphi_{t_1}(y)$, $\pi_x \circ P_{t_1} \circ \pi_y(B_2^{cs})$ and to the box containing B_1^{cu} , the iterate $\pi_x P_{t_1} \pi_y(B_2^{cs})$ can not intersect the \mathcal{E} -boundary of B_1^{cu} . This implies that $\pi_x P_{t_1} \pi_y(B_2^{cs})$ is contained in B_1^{cu} , hence $B_2^{cs} \subset B_1^{cs}$.

The proof of Theorem 5.3 is now complete. \square

6 Uniform hyperbolicity

In this section we prove Theorem C (see section 6.3).

Standing assumptions. In the whole section, (\mathcal{N}, P) is a C^2 local fibered flow over a topological flow (K, φ) which is not a periodic orbit and π is a C^2 -identification compatible with (P_t) on an open set U as in Definition 3.4 such that:

- (B1) There is a dominated splitting $\mathcal{N} = \mathcal{E} \oplus \mathcal{F}$ and the fibers of \mathcal{E}, \mathcal{F} are one-dimensional.
- (B2) \mathcal{E} is uniformly contracted on an open set V containing $K \setminus U$.
- (B3) \mathcal{E} is uniformly contracted on any compact invariant proper subset of K .
- (B4) \mathcal{E} is topologically contracted.

The main result of this section is Proposition 6.1, which is proved in the next two sections.

Proposition 6.1. *Under the standing assumptions above, for any ergodic invariant measure μ whose support is K , if the Lyapunov exponent of μ along \mathcal{F} is positive, then the Lyapunov exponent of μ along \mathcal{E} is negative.*

Consider a measure μ as in the statement of the proposition. We recall that $K = \text{supp}(\mu)$ is not a periodic orbit. In particular the assumptions of Theorem 5.3 are satisfied. One will choose $x \in K \setminus \overline{V}$, some β_x small (to be precised later) and consider a Markovian box $R \subset B(0_x, \beta_x)$.

The proof is divided into two cases: the non-minimal case and the minimal case.

6.1 The non-minimal case

In this section, we will prove Proposition 6.1 when **the dynamics on K is not minimal**. Since $K = \text{supp}(\mu)$, the dynamics on K is transitive. One can thus fix a non-periodic point $x \in K \setminus \overline{V}$ whose orbit is dense in K and reduce r_0 so that:

- the ball $U(x, r_0) \subset K$ centered at x with radius r_0 is contained in U ,
- the maximal invariant set in $K \setminus U(x, r_0)$ is non-empty.

We still denote $\mu_x := (\pi_x)_*(\mu|_{U(x, r_0)})$.

6.1.1 Notations, choices of constants

a – The constant β_x , the box R , the sets \widehat{R} and W . One chooses some constants $\beta_x > 0$ and $T_x \geq 1$ which satisfy Lemma 3.33. One will also assume that β_x smaller than β_S in Lemma 3.27. Theorem 5.3 associates to β_x a box $R \subset B(0_x, \beta_x) \subset \mathcal{N}_x$ whose interior has positive μ_x -measure by Theorem 5.3.

Notation. For any box $B \subset \mathcal{N}_x$, we denote by \widehat{B} the following open subset of K :

$$\widehat{B} := \{y \in K, d(x, y) < r_0/2 \text{ and } \pi_x(y) \in \text{Interior}(B)\}.$$

Let W be the set of points z such that $\varphi_s(z) \notin \widehat{R}$ for any $s \in [0, 1]$.

b – The constants $T_{\mathcal{F}}, C_{\mathcal{E}}, \lambda_{\mathcal{E}}, C_{\mathcal{F}}, \lambda_{\mathcal{F}}$. Note that \mathcal{E} is uniformly contracted on the set $W' := \bigcup_{s \in [0,1]} \varphi_s(W)$: indeed this set is disjoint from the open set \widehat{R} and by our choice of r_0 it contains a non-empty compact invariant proper set $K' \subset K$; if \mathcal{E} is not uniformly contracted on the set W' , one gets a contradiction with our assumption (B3). Proposition 3.34 can thus be applied to W and gives some $C_{\mathcal{E}}^0, \lambda_{\mathcal{E}}$ and $T_{\mathcal{F}} \geq T_x$.

One sets $C_{\mathcal{E}} = C_{\mathcal{E}}^0 \max_{-1 \leq s \leq 1} \|DP_t\|^2$. Consequently a piece of orbit $(y, \varphi_t(y))$ is $(C_{\mathcal{E}}, \lambda_{\mathcal{E}})$ -hyperbolic for \mathcal{E} , once there exists $s_0, s_1 \in [-1, 1]$ and a piece of orbit $(\varphi_{-k}(z), z)$ satisfying the assumptions of Proposition 3.34 and $y = \varphi_{-k+s_1}(z)$, $\varphi_t(y) = \varphi_{s_2}(z)$.

c – Hyperbolicity for \mathcal{E} in $K \setminus \widehat{R}$. By (B3), the bundle \mathcal{E} is uniformly contracted outside \widehat{R} ; one can thus relax again $C_{\mathcal{E}}, \lambda_{\mathcal{E}}$ so that any $y \in K$ and $t > 0$ satisfy:

$$\forall s \in [1/2, t - 1/2], \varphi_s(y) \notin \widehat{R} \Rightarrow \|DP_t|_{\mathcal{E}(y)}\| \leq C_{\mathcal{E}} \lambda_{\mathcal{E}}^{-t}.$$

d – Hyperbolicity for \mathcal{F} . As in Section 3.3.1, λ is associated to the 2-domination $\mathcal{E} \oplus \mathcal{F}$.

Let $\lambda_{\mathcal{F}} := \lambda^{1/2}$. There is $C_{\mathcal{F}} > 0$ with the following property:

If $(y, \varphi_{t-s}(y))$ is a $(2T_{\mathcal{F}}, \lambda_{\mathcal{F}})$ -Pliss string for some $s \in [0, 1]$, then $(y, \varphi_t(y))$ is $(C_{\mathcal{F}}^{1/2}, \lambda_{\mathcal{F}})$ -hyperbolic for \mathcal{F} .

e – The constants $\lambda'_{\mathcal{E}}, \beta_{box}$. We need weaker constants $C'_{\mathcal{E}}, \lambda'_{\mathcal{E}}$ for the hyperbolicity along \mathcal{E} . We first set $\lambda'_{\mathcal{E}} = \lambda_{\mathcal{E}}^{1/2}$ and then apply the following lemma: fixing a transition (x_0, t_0) , it defines for points $y \in \widehat{B}^{cs}$ a piece of orbit $(y, \varphi_{t(y)}(y))$ shadowing $(x_0, \varphi_{t_0}(x_0))$.

Lemma 6.2. *There exists $\beta_{box} > 0$ such that if B_1, \dots, B_k, t_{box} are boxes and the constant associated to $R, C_{\mathcal{F}}, \lambda_{\mathcal{F}}, \beta_{box}$ by Theorem 5.3, then the following holds for some $C'_{\mathcal{E}} > 0$.*

Let (x_0, t_0) be a transition with $t_0 > 2t_{box}$ such that $(x_0, \varphi_n(x_0))$ is $T_{\mathcal{F}}$ -Pliss, where $n \in [t_0 - 1, t_0]$; let B^{cs}, B^{cu} be the associated sub-boxes, and $y \in \widehat{B}^{cs}$. Then there exist $\theta \in \text{Lip}_2$ and $t(y)$ such that:

1. $\varphi_{t(y)}(y) \in \widehat{B}^{cu}$,
2. $|\theta(0)| \leq 1/4, |\theta(t_0) - t(y)| \leq 1/4$,
3. $d(\varphi_t(x_0), \varphi_{\theta(t)}(y)) < r_0/2$ for $t \in [-1, t_0 + 1]$,
4. $(y, \varphi_{t(y)}(y))$ is a $(2T_{\mathcal{F}}, \lambda_{\mathcal{F}})$ -Pliss string.

If $(x_0, \varphi_{t_0}(x_0))$ is $(C_{\mathcal{E}}, \lambda_{\mathcal{E}})$ -hyperbolic for \mathcal{E} , then $(y, \varphi_{t(y)}(y))$ is $(C'_{\mathcal{E}}, \lambda'_{\mathcal{E}})$ -hyperbolic for \mathcal{E} .

From the items 1 and 4 of this Lemma and the choice of $C_{\mathcal{F}}, (y, t(y))$ is a transition. The associated boxes are B^{cs} and B^{cu} . Indeed, the items 2 and 3 together with the item 5 of Theorem 5.3 imply that the associated center-unstable box coincides with B^{cu} . By the Global invariance, the center-stable box $\pi_x \circ P_{-t(y)} \pi_{\varphi_{t_0}(y)}(B^{cu})$ coincides with B^{cs} .

Proof. For $\lambda' \in (1, \lambda_{\mathcal{E}}^{1/4})$, Lemma 3.29 gives C', δ, ρ . We may take $\rho \in (0, 1/3)$. The Global invariance then associates to δ, ρ some constants β, r . By the Local injectivity, we can take $\beta_{box} \in (0, \beta)$ smaller such that for any $x, y \in U$ that are r_0 -close to x and satisfy $d(\pi_x(y), \pi_x(x_0)) \leq \beta_{box}$, there exists $s \in [-1/4, 1/4]$ such that $d(x, \varphi_s(y)) < r$.

By Lemma 3.33 (with the constants β_x and $T_{\mathcal{F}} \geq T_x$ introduced in paragraphs (a) and (b) above), we obtain $\theta \in \text{Lip}_2$ such that $|\theta(0)| \leq 1/4$ and item 3 is satisfied; moreover $(y, \varphi_{\theta(y)+a}(y))$ is a $(2T_{\mathcal{F}}, \lambda^{1/2})$ -Pliss string for any $a \in [-1, 1]$. By the Local injectivity, there exists $t(y) \in [\theta(t_0) - 1/4, \theta(t_0) + 1/4]$ such that $d(\varphi_{t(y)}(y), x) < r_0/2$ and $\pi_x \circ \varphi_{t(y)}(y) = \pi_x \circ \varphi_{\theta(t_0)}(y)$. In

particular item 2 holds. Moreover $\pi_{\varphi_{t_0}(x_0)} \circ \varphi_{\theta(t_0)}(y) = P_{t_0} \circ \pi_{x_0}(y) \in P_{t_0} \circ \pi_{x_0}(B^{cs})$. Its projection by π_x belongs to $B^{cu} = \pi_x \circ P_{t_0} \circ \pi_{x_0}(B^{cs})$, so $\pi_x \circ \varphi_{\theta(t_0)}(y) \in B^{cu}$ giving the first item.

Let us assume now that $(x_0, \varphi_{t_0}(x_0))$ is $(C_{\mathcal{E}}, \lambda_{\mathcal{E}})$ -hyperbolic for \mathcal{E} and consider $\sigma \in [t_{box}, t_0 + 1/4]$ such that

- $\varphi_{\theta(\sigma)}(y) \in \widehat{R}$,
- $\varphi_{\theta(s)}(y) \notin \widehat{R}$ for $s \in [t_{box}, t_0]$ satisfying $\theta(s) \leq \theta(\sigma) - 1/2$.

From the property stated at paragraph (c) above, we have for any $s \in [t_{box}, \sigma]$

$$\|DP_{\theta(s)-\theta(t_{box})}|\mathcal{E}(\varphi_{\theta(t_{box})}(y))\| \leq C_{\mathcal{E}}\lambda_{\mathcal{E}}^{-(\theta(s)-\theta(t_{box}))}.$$

Since $\theta(t_{box}) < 2t_{box}$, there exists $C_1 > 0$ independent from x_0, t_0, y such that for any $s \in [0, \sigma]$,

$$\|DP_{\theta(s)}|\mathcal{E}(y)\| \leq C_1\lambda_{\mathcal{E}}^{-\theta(s)}. \quad (13)$$

From Local injectivity (recalled above) and since $\text{Diam}(P_s \circ \pi_{x_0}(B^{cu})) < \beta_{box}$ for $s \in [t_{box}, t_0]$ (by item 4 of Theorem 5.3), there exists $\varepsilon \in [-1/4, 1/4]$ such that $d(\varphi_{\theta(\sigma)+\varepsilon}(y), \varphi_{\sigma}(x_0)) < r$. The Global invariance gives $\theta' \in \text{Lip}_{1+\rho}$, such that for each $s \in [t_{box} - 1, t_0 + 1]$ one has $d(\varphi_{\theta'(s)}(y), \varphi_s(x_0)) < \delta$ and $\theta'(\sigma) = \theta(\sigma) + \varepsilon$.

Lemma 3.29 now gives for each $s \in [t_{box} - 1, t_0 + 1]$,

$$\|DP_{\theta'(s)-\theta'(t_{box})}|\mathcal{E}(\varphi_{\theta'(t_{box})}(y))\| \leq C'\lambda'^{s-t_{box}}\|DP_{s-t_{box}}|\mathcal{E}(\varphi_{t_{box}}(x_0))\|.$$

Since $(x_0, \varphi_{t_0}(x_0))$ is $(C_{\mathcal{E}}, \lambda_{\mathcal{E}})$ -hyperbolic for \mathcal{E} , since θ' is 3/2-bi-Lipschitz, and since $\lambda' < \lambda_{\mathcal{E}}^{1/4}$, one gets $C_2 > 0$ (depending on t_{box} , not on x_0, t_0, y) such that for any $s \in [t_{box} - 1, t_0 + 1]$,

$$\|DP_{\theta'(s)-\theta'(t_{box})}|\mathcal{E}(\varphi_{\theta'(t_{box})}(y))\| \leq C_2\lambda'^{s-t_{box}}C_{\mathcal{E}}\lambda_{\mathcal{E}}^{-(s-t_{box})} \leq C_2C_{\mathcal{E}}\lambda_{\mathcal{E}}^{-(\theta'(s)-\theta'(t_{box}))/2}. \quad (14)$$

Combining (13) and (14), one deduces that $(y, \varphi_{\theta(t_0)}(y))$ is $(C'_{\mathcal{E}}, \lambda'_{\mathcal{E}})$ -hyperbolic for \mathcal{E} for some constant $C'_{\mathcal{E}}$, provided $\theta'(t_{box} - 1) \leq \theta(\sigma)$ and $\theta'(t_0 + 1) \geq \theta(y)$.

Since θ' is 2-bi-Lipschitz, one gets $\theta'(t_{box} - 1) \leq \theta'(\sigma) - 1/2 = \theta(\sigma) + \varepsilon - 1/2 < \theta(\sigma)$. One can apply Proposition 3.6 to $\varphi_{\theta(\sigma)}(y)$, the reparametrization $\theta' \circ \theta^{-1}$ and the interval $[\theta(\sigma), \theta(t_0)]$. Since $|\theta'(\sigma) - \theta(\sigma)| < 2$, one gets $\theta'(t_0) + 1/2 \geq \theta(t_0)$. Since θ' is 4/3-bi-Lipschitz, this gives $\theta'(t_0 + 1) \geq \theta'(t_0) + 3/4 \geq \theta(t_0) + 1/4 \geq \theta(y)$ and concludes the proof. \square

f – The sub-boxes B_1, \dots, B_k and the constants $t_{box}, \Delta_{box}, C'_{\mathcal{E}}$. Finally we apply Theorem 5.3 to $R, C_{\mathcal{F}}, \lambda_{\mathcal{F}}, \beta_{box}$ and obtain the sub-boxes B_1, \dots, B_k and the constants t_{box}, Δ_{box} that we fix now. Lemma 6.2 gives $C'_{\mathcal{E}}$.

6.1.2 Existence of large hyperbolic returns

Since we have to prove that the Lyapunov exponent of μ along \mathcal{E} is negative, one can reduce to the case where the following condition holds:

(B5) The Lyapunov exponent of μ along \mathcal{E} is larger than $-\log(\lambda_{\mathcal{E}})$.

In the following we will say that a point $z \in K$ is called *regular* if :

- the orbit of z equidistributes towards μ , i.e. $\frac{1}{t} \int_0^t \delta_{\varphi_s(z)} ds \rightarrow \mu$ as $t \rightarrow \pm\infty$.
- For any iterate $\varphi_t(z)$, if $d(\varphi_t(z), x) < r_0$, then $\pi_x(\varphi_t(z))$ is not contained in the boundary of R , nor of any box B_i , $1 \leq i \leq k$.

By Birkhoff ergodic theorem and since the boundaries of the boxes R and B_i have zero measure (for $\mu_x = (\pi_x)_*(\mu)$), the set of regular points has full measure for μ .

Lemma 6.3. *For any $T_0 > 0$, there exists a regular point x_0 and $t_0 > T_0$ such that*

- both x_0 and $\varphi_{t_0}(x_0)$ are in \widehat{R} ,
- $(x_0, \varphi_n(x_0))$ is $T_{\mathcal{F}}$ -Pliss for some $n \in [t_0 - 1, t_0]$.

Proof. Let us take a regular point y . We have $\omega(x) = \alpha(x) = K$. By assumption the maximal invariant set outside \widehat{R} is a non-empty compact invariant proper set K_0 of K . One can assume that y is very close to K_0 . Thus, backward iterates $\varphi_{t_1}(y)$ and forward iterate $\varphi_{t_2}(y)$ in \widehat{R} occur for t_1 and t_2 large: clearly, one can choose y such that $t_2 - t_1 > T_0 + 1$. Choosing t_1, t_2 close to their infimum values, one furthermore gets that $\varphi_t(y) \notin \widehat{R}$ for $t \in (t_1 + 1/4, t_2 - 1/4)$.

Let $x_0 := \varphi_{t_1}(y)$. Note that x_0 is not $(C_{\mathcal{E}}, \lambda_{\mathcal{E}})$ -hyperbolic for \mathcal{E} : since x_0 is regular, its forward orbit equidistributes on the measure μ and this would imply that the Lyapunov exponent of μ along \mathcal{E} is less than or equal to $-\log(\lambda_{\mathcal{E}})$, contradicting the assumption (B5). By Proposition 3.34 and the choice of constants in 6.1.1(b), there exists a forward iterate $\varphi_n(x_0) \notin W$, $n \geq 1$, such that $(x_0, \varphi_n(x_0))$ is a $T_{\mathcal{F}}$ -Pliss string for \mathcal{F} . By definition of W , there is $t_0 \in [n, n+1]$ such that $\varphi_{t_0}(x_0) \in \widehat{R}$. By our choice of t_1, t_2 , one has $t_0 \geq t_2 - 1/2 - t_1 > T_0$. \square

6.1.3 Contraction at returns

Let (x_0, t_0) be given by Lemma 6.3 for $T_0 > 2t_{box}$. Consider sub-boxes B_{i_0}, B_{j_0} such that

$$\pi_x(x_0) \in \text{Interior}(B_{i_0}), \quad \pi_x(\varphi_{t_0}(x_0)) \in \text{Interior}(B_{j_0}).$$

We get a transition (x_0, t_0) between B_{i_0} and B_{j_0} . By Theorem 5.3, one thus gets a center-stable sub-box $B^{cs} \subset B_{i_0}$ and a center-unstable sub-box $B^{cu} \subset B_{j_0}$ as in the statement of this theorem.

Lemma 6.4. *If t_0 is large enough, there exists $\lambda_* > 1$ (depending on t_0) such that for any regular $y \in \widehat{B}^{cs}$, there is $\tau > t_{box}$ satisfying $\varphi_{\tau}(y) \in \widehat{B}^{cs}$ and*

$$\|DP_{\tau}|_{\mathcal{E}(y)}\| \leq \lambda_*^{-\tau}.$$

The proof of this lemma breaks into 5 steps.

Step 1. Definition of the times $\sigma < t(y) \leq \tau$. For any regular $y \in \widehat{B}^{cs}$, Lemma 6.2 gives a time $t(y)$ such that $\varphi_{t(y)}(y) \in \widehat{B}^{cu}$ and $(y, \varphi_{t(y)}(y))$ shadows the piece of orbit $(x_0, \varphi_{t_0}(x_0))$.

The forward orbit of y is dense in K , hence there is $\tau \geq t(y)$ such that

$$\varphi_{\tau}(y) \in \widehat{B}^{cs}, \text{ but } \varphi_s(y) \notin \widehat{B}^{cs} \text{ for any } s \in (t(y), \tau - 1).$$

We also introduce a return time $\sigma \in [0, t(y) - 1]$ (possibly equal to 0) such that

$$\varphi_{\sigma}(y) \in \widehat{B}^{cs}, \text{ but } \varphi_s(y) \notin \widehat{B}^{cs} \text{ for any } s \in (\sigma + 1, t(y) - 1).$$

Step 2. Definition of the times $t_1 < t_2 < \dots < t_\ell$. We now introduce intermediate times between $\sigma + 1$ and $\tau - 1$. We first set

$$t_1 = t(\varphi_\sigma(y)).$$

By applying Lemma 6.2 twice, the orbit segment $(y, \varphi_{t_1}(y))$ is $(C'_\mathcal{E}{}^2, \lambda'_\mathcal{E})$ -hyperbolic for \mathcal{E} . Let $C_2 = \max_{0 \leq t \leq 2} \|DP_t\| \lambda'_\mathcal{E}{}^2$. From Lemma 6.2, we get

$$\tau \geq t(y) \geq \frac{1}{2}t_0 - \frac{1}{4}.$$

If $t_1 + 2 \geq \tau$, then provided t_0 has been chosen large enough one gets

$$\|DP_\tau|_{\mathcal{E}(y)}\| \leq C_2 C'_\mathcal{E}{}^2 \lambda'_\mathcal{E}{}^{-\tau} \leq C_2 C'_\mathcal{E}{}^2 \lambda'_\mathcal{E}{}^{-\frac{1}{2}(\frac{t_0}{2}-1/4)} \lambda'_\mathcal{E}{}^{-\tau/2} \leq \lambda'_\mathcal{E}{}^{-\tau/2}.$$

Hence the conclusion of Lemma 6.4 holds in this case with $\lambda_* = \lambda'_\mathcal{E}{}^{1/2}$. A similar discussion holds when $\tau \leq t(y) + 2$. Thus, without loss of generality, we can assume that:

$$t_1 + 2 < \tau \text{ and } t(y) + 2 < \tau.$$

Sublemma 6.5. *There exists a sequence of times $\{t_m\}_{m=2}^\ell$ in $[t_1 + 1, \tau - 1]$ such that:*

- $\varphi_{t_m}(y) \in \widehat{R}$ and $(\varphi_\sigma(y), \varphi_{t_m}(y))$ is $(C_\mathcal{F}, \lambda_\mathcal{F})$ -hyperbolic for \mathcal{F} .
(Equivalently $(\varphi_\sigma(y), t_m - \sigma)$ is a transition.)
- $t_m \geq t_{m-1} + 1$ and $(\varphi_{t_{m-1}}(y), \varphi_{t_m}(y))$ is $(C_\mathcal{E}, \lambda_\mathcal{E})$ -hyperbolic for \mathcal{E} .
- $(\varphi_{t_\ell}(y), \varphi_\tau(y))$ is $(C_\mathcal{E}, \lambda_\mathcal{E})$ -hyperbolic for \mathcal{E} .

Proof. We define inductively the increasing sequence of integers $\{n_m\}_{m=1}^\ell$ such that:

- $n_1 = 0$ and for any $2 \leq m \leq \ell$, the piece of orbit $(\varphi_{t_1}(y), \varphi_{t_1+n_m}(y))$ is a $T_\mathcal{F}$ -Pliss string, $n_m - n_{m-1} \geq 2$ and $\varphi_{t_1+n_m}(y) \notin W$;
- for any integer $0 \leq n \leq \tau(y) - t_1(y) - 2$ such that neither n , nor $n-1$ belong to $\{n_1, \dots, n_\ell\}$, then either $\varphi_{t_1+n}(y) \in W$ or $(\varphi_{t_1}(y), \varphi_{t_1+n}(y))$ is not a $T_\mathcal{F}$ -Pliss string.

By definition of W , there exists $t_m \in [t_1 + n_m, t_1 + n_m + 1]$ such that $\varphi_{t_m}(y) \in \widehat{R}$. Note that we have $t_m \geq t_{m-1} + 1$ and $t_\ell + 1 \leq \tau$.

By our choice of $C_\mathcal{F}, \lambda_\mathcal{F}$, the piece of orbit $(\varphi_{t_1}(y), \varphi_{t_m}(y))$ is $(C_\mathcal{F}^{1/2}, \lambda_\mathcal{F})$ -hyperbolic for \mathcal{F} . By Lemma 6.2, $(\varphi_\sigma(y), \varphi_{t_1}(y))$ is a $(2T_\mathcal{F}, \lambda_\mathcal{F})$ -Pliss string, hence is also $(C_\mathcal{F}^{1/2}, \lambda_\mathcal{F})$ -hyperbolic for \mathcal{F} . Consequently $(\varphi_\sigma(y), \varphi_{t_m}(y))$ is $(C_\mathcal{F}, \lambda_\mathcal{F})$ -hyperbolic for \mathcal{F} .

By our choice of n_m , any integer n with $n_{m-1} + 2 \leq n < n_m$ either belongs to W or satisfies that $(\varphi_{t_1}(y), \varphi_{t_1+n}(y))$ is not $T_\mathcal{F}$ -Pliss. Proposition 3.34 and the choice of $(C_\mathcal{E}, \lambda_\mathcal{E})$ implies that $(\varphi_{t_1+n_{m-1}}(y), \varphi_{t_1+n_m}(y))$ and $(\varphi_{t_{m-1}}(y), \varphi_{t_m}(y))$ are $(C_\mathcal{E}, \lambda_\mathcal{E})$ -hyperbolic for \mathcal{E} . This gives the second item. The third item is obtained similarly. \square

Step 3. Construction of center-unstable boxes associated to the times t_m . By Sublemma 6.5, $(\varphi_\sigma(y), t_m - \sigma)$ is a Markovian transition between boxes in $\{B_1, B_2, \dots, B_k\}$ for any $1 \leq m \leq \ell$. By Theorem 5.3 it defines a center-stable sub-box B_m^{cs} and a center-unstable sub-box B_m^{cu} . Moreover the distortion of B_m^{cu} is bounded by the constant Δ_{box} .

Sublemma 6.6. *The interiors of the boxes B_m^{cu} , for $t_{box} < m \leq \ell$, are mutually disjoint.*

Proof. Let us first notice that if $m > t_{box}$, the center-stable sub-box B_m^{cs} is contained in B^{cs} : indeed, let us consider the transitions $(\varphi_\sigma(y), \varphi_{t_1}(y))$ and $(\varphi_\sigma(y), \varphi_{t_m}(y))$. We have $t_1 - \sigma > t_{box}$ and $t_m - t_1 > t_{box}$. Moreover, the boxes associated to the first transition are B^{cs}, B^{cu} (as explained after the Lemma 6.2). Theorem 5.3, item 6, implies that B_m^{cs} is contained in $B^{cs} = B_1^{cs}$.

Assume by contradiction that the interiors of B_i^{cu} and B_j^{cu} , for $i \neq j$ larger than t_{box} , intersect. Up to exchange i and j , the item 5 of Theorem 5.3 gives $\theta \in \text{Lip}_2$ such that

- $d(\varphi_s(y), \varphi_{\theta(s)}(y)) < r_0/2$, for any $s \in [\sigma, t_i] \cap \theta^{-1}([\sigma, t_j])$,
- $|\theta(t_i) - t_j| \leq 1/2$ and $\theta(\sigma) \geq \sigma - 1$.

Claim. $\theta(\sigma) > \sigma + 2$.

Proof. By Proposition 3.6, $\theta(\sigma) \in [\sigma - 1, \sigma + 2]$ implies $|\theta(t_i) - t_i| < 1/2$. This gives $|t_i - t_j| < 1$ and this contradicts the definition of the sequence (t_m) since $t_{m+1} - t_m \geq 1$ for any m . \square

Since $B_i^{cs} \subset B^{cs}$, the image $\pi_{\varphi_\sigma(y)}(B_i^{cs}) = P_{-(t_i-\sigma)} \circ \pi_{\varphi_{t_i}(y)}(B_i^{cu})$ by π_x is contained in B^{cs} . Since $\pi_x \circ \varphi_{t_j}(y)$ belongs to $B_j^{cu} \subset B_i^{cu}$, one gets

$$\pi_x \circ P_{-(t_i-\sigma)} \circ \pi_{\varphi_{t_i}(y)}(\varphi_{t_j}(y)) \in B^{cs}.$$

Using the Global invariance this gives

$$\pi_x \circ P_{-(\theta(t_i)-\theta(\sigma)(y))} \circ \pi_{\varphi_{\theta(t_i)}(y)}(\varphi_{t_j}(y)) \in B^{cs}.$$

Since $|t_j - \theta(t_i)| \leq 1/2$, the Local invariance gives $0_{\varphi_{\theta(t_i)}(y)} = \pi_{\varphi_{\theta(t_i)}(y)}(\varphi_{t_j}(y))$, hence $\pi_x \circ \varphi_{\theta(\sigma)}(y) \in B^{cs}$. The local injectivity gives s with $|\theta(\sigma) - s| \leq 1/4$ such that $\varphi_s(y) \in \widehat{B}^{cs}$.

Claim. We have $t(y) + 1 < s < \tau - 1$.

Proof. Since $\theta(\sigma) > \sigma + 2$, we have $\sigma + 1 < s$. By definition of σ , one gets $s \geq t(y) - 1$.

Note that $\pi_x(\varphi_s(y)) = \pi_x(\varphi_{t(y)}(t))$ when $s \in [t(y) - 1, t(y) + 1]$, hence $\varphi_{t(y)}(y) \in \widehat{B}^{cs}$; this gives $\tau = t(y)$ by definition and contradicts the assumption $\tau > t(y) + 1$.

Since $\theta(\sigma) \leq \theta(t_i) \leq t_j + 1/2$, one gets $s \leq t_j + 3/4$. Since $t_\ell + 2 < \tau$, we have $s < \tau - 1$. \square

We have thus obtained a time s which contradicts the definition of τ . This concludes the proof of Sublemma 6.6. \square

Step 4. Summability. Let $J^{cs}(y) = \mathcal{W}^{cs}(y) \cap \pi_y(B^{cs})$. In each sub-box B_j of R , we choose a C^1 -curve γ_j tangent to $\mathcal{C}^\mathcal{E}$ with endpoints in $\partial^{cu}B_j$. Set

$$L_B = \sum_{1 \leq j \leq k} |\gamma_j|.$$

It only depends on R and B_1, \dots, B_k , but not on the points x_0, y .

Sublemma 6.7. We have

$$\sum_{i=0}^{[\tau]} |P_i(J^{cs}(y))| \leq C_{sum} := \frac{2C'_\mathcal{E} \lambda'_\mathcal{E}}{\lambda'_\mathcal{E} - 1} \Delta_{box} L_B (1 + t_{box}).$$

Proof. For each $1 \leq m \leq \tau$ we have $|P_{t_m}(J^{cs}(y))| \leq \Delta_{box} L_B$. Moreover $P_{t_m}(J^{cs}(y))$ is a curve tangent to $\mathcal{C}^\mathcal{E}$ and crosses B_m^{cu} . Since the interiors of $\{B_m^{cu}, t_{box} < m \leq \ell\}$ are mutually disjoint (Lemma 6.6) and are center-unstable sub-boxes of B_1, \dots, B_k which have distortion bounded by Δ_{box} , we have that

$$\sum_{1 \leq m \leq \ell} |P_{t_m}(J^{cs}(y))| \leq \Delta_{box} L_B (1 + t_{box}).$$

By Sublemma 6.5, $(\varphi_{t_m}(y), \varphi_{t_{m+1}}(y))$ is $(C_\mathcal{E}, \lambda_\mathcal{E})$ -hyperbolic for \mathcal{E} . Thus, we have

$$\sum_{t_m \leq i \leq t_{m+1}} |P_i(J^{cs}(y))| \leq \frac{C_\mathcal{E} \lambda_\mathcal{E}}{\lambda_\mathcal{E} - 1} |P_{t_m}(J^{cs}(y))|.$$

A similar estimate holds for integers i in $[t_\ell, \tau]$. Hence

$$\sum_{t_1 \leq i \leq \tau} |P_i(J^{cs}(y))| \leq \frac{C_\mathcal{E} \lambda_\mathcal{E}}{\lambda_\mathcal{E} - 1} \Delta_{box} L_B (1 + t_{box}).$$

We have shown previously that $(y, \varphi_{t_1}(y))$ is $(C'_\mathcal{E}, \lambda'_\mathcal{E})$ -hyperbolic for \mathcal{E} , hence

$$\sum_{0 \leq i < t_1} |P_i(J^{cs}(y))| \leq \frac{C'^2_\mathcal{E} \lambda'_\mathcal{E}}{\lambda'_\mathcal{E} - 1} \Delta_{box} L_B.$$

The estimate of the sublemma follows from these two last inequalities. \square

Step 5. End of the proof of Lemma 6.4. By item 4 of Theorem 5.3, for any $0 \leq s \leq \tau$ one has $P_s(J^{cs}(y)) \subset B(0_{\varphi_s(y)}, \beta_S)$. Lemma 3.27 associates to C_{Sum} a constant $C_S > 1$, independent from x_0, t_0 and gives

$$\|DP_\tau|_{\mathcal{E}(y)}\| \leq C_S \frac{|P_\tau J^{cs}(y)|}{|J^{cs}(y)|}.$$

By construction $\tau \geq \frac{t_0}{2} - 1/4$. Moreover, $|J^{cs}(y)|$ is bounded away from zero independently from x_0, t_0 . The topological hyperbolicity of \mathcal{E} ensures that $|P_\tau J^{cs}(y)|$ is arbitrarily small if τ is large. As a consequence, if t_0 is large enough, for any regular $y \in \widehat{B}^{cs}$, one gets

$$\|DP_\tau|_{\mathcal{E}(y)}\| \leq \frac{1}{2}.$$

By assumption (B3), \mathcal{E} is uniformly contracted on the maximal invariant set in $K \setminus \widehat{B}^{cs}$. The time $t(y)$ is bounded uniformly in y and $\varphi_s(y)$ does not meet \widehat{B}^{cs} for $s \in (t(y), \tau - 1)$. Hence there exists $C_B, \lambda_B > 1$ (depending on x_0, t_0) such that for any regular $y \in \widehat{B}^{cs}$,

$$\|DP_\tau|_{\mathcal{E}(y)}\| \leq C_B \lambda_B^{-\tau}.$$

Choosing $\lambda_* > 1$ close to 1 one gets for any $t > 0$,

$$\min(1/2, C_B \lambda_B^{-t}) \leq \lambda_*^{-t},$$

which gives the estimate of Lemma 6.4. \square

6.1.4 Proof of Proposition 6.1 in the non-minimal case

We can now conclude the proof of Proposition 6.1 when the dynamics on K is non-minimal. For any regular point $y \in \widehat{B}^{cs}$, we have obtained a contracting return $\tau(y)$. This allows to define an increasing sequence of times $\{\tau_n\}_{n \in \mathbb{N}}$ such that $\tau_0 = \tau(y)$ and $\tau_{n+1} = \tau_n + \tau(\varphi_{\tau_n}(y))$. By Lemma 6.4, we have for any $n \geq 0$

$$\|DP_{\tau_n}|_{\mathcal{E}(y)}\| \leq \lambda_*^{-\tau_n}.$$

Since y is regular, $\frac{1}{\tau_n} \log(\|DP_{\tau_n}|_{\mathcal{E}(y)}\|)$ converges as $n \rightarrow +\infty$ to the Lyapunov exponent of μ along \mathcal{E} . Consequently this Lyapunov exponent is smaller or equal to $-\log(\lambda_*)$. Hence it is negative as announced. \square

6.2 The minimal case

In this section, we will continue to prove Proposition 6.1, now assuming that **the dynamics on K is minimal**. We will apply a local version of the result of Pujals and Sambarino [PS₁].

Theorem 6.8. *Assume that $f : W_1 \rightarrow W_2$ is a C^2 diffeomorphism, where $W_1, W_2 \subset \mathbb{R}^2$ are open sets, and that there is a compact invariant set $\Lambda \subset W_1 \cap W_2$ of f such that:*

- *every periodic point in Λ is a hyperbolic saddle,*
- *Λ admits a dominated splitting $T_\Lambda \mathbb{R}^2 = E \oplus F$,*
- *Λ does not contain a circle tangent to E or F which is invariant by an iterate of f ,*

then Λ is hyperbolic.

Note that Pujals-Sambarino stated their theorem for global diffeomorphisms of a compact surface, but their proof gives also the local result above. It is also obtained in [CPS].

Our goal now is to reduce the minimal case to Theorem 6.8 by introducing a local surface diffeomorphism and an invariant compact set Λ .

Let us consider some r small such that the “No small period” assumption holds for some $\varkappa < 1/2$. As before one chooses a point $x \in K \setminus \overline{V}$, some β_x , and a box $R \subset B(0_x, \beta_x)$ given by Theorem 5.3. We introduce the set

$$\widehat{R} = \{y \in K, d(y, x) < r_0/2 \text{ and } \pi_x(y) \in R\}.$$

Assuming that β_x has been chosen small enough, the Local injectivity associates to any $y \in \widehat{R}$, a point $y' \in \widehat{R}$ such that $d(y', x) < r/2$ and $\pi_x(y) = \pi_x(y')$. Moreover $y' = \varphi_t(y)$ for some $t \in [-1/4, 1/4]$.

The set Λ . We introduce the following set:

$$\Lambda := \pi_x(\widehat{R}) = \{u \in R, \exists y \in K, d(x, y) < r_0/2 \text{ and } u = \pi_x(y)\}.$$

Note that, one can choose the points y in the definition of Λ to be $r/2$ -close to x and in particular to satisfy $d(x, y) \leq r_0/3$.

Lemma 6.9. *Λ is compact and contained in the interior of R .*

Proof. Indeed, let us consider $\{u_n\}_{n \in \mathbb{N}}$ in Λ such that $\lim_{n \rightarrow \infty} u_n = u$. We take $y_n \in K$ that is $r_0/3$ -close to x such that $\pi_x(y_n) = u_n$. Taking a subsequence if necessary, we assume that $y = \lim_{n \rightarrow \infty} y_n$. This point is $r_0/3$ -close to x and by continuity of the identification $\pi_x(y) = u$. Hence $u \in \Lambda$, proving that Λ is compact.

Since K is not periodic and is minimal, K does not contain periodic orbits. By property 1 of Theorem 5.3, one deduces that for any point y which is r_0 -close to x , the projection $\pi_x(y)$ is disjoint from the boundary ∂R . Hence $\Lambda = \pi_x(\widehat{R})$ is contained in the interior of R . \square

The return map f on Λ . For any $u \in \Lambda$, one defines $f(u)$ as follows: consider $y \in \widehat{R}$ such that $\pi_x(y) = u$ and choose the smallest $t \geq 1$ such that $d(\varphi_t(y), x) \leq r_0/3$ and $\pi_x(\varphi_t(y)) \in \Lambda$ (such a t exists because K is minimal). We then define $f(u) = \pi_x(\varphi_t(y))$.

Lemma 6.10. *f is well defined.*

Proof. We have to check that the definition of $f(u)$ does not depend on the choice of y . So we consider $y, y' \in \widehat{R}$ such that $\pi_x(y) = \pi_x(y')$ and the minimal times t, t' as in the previous definition. By the Local injectivity, there exists $s_0 \in [-1/4, 1/4]$ such that $y_0 := \varphi_{s_0}(y)$ is $r/2$ -close to x and $\pi_x(y) = \pi_x(y_0)$. One builds similarly s'_0 and y'_0 . Then y_0, y'_0 are r_0 close to each other and satisfy $\pi_x(y_0) = \pi_x(y'_0)$ so that by the Local injectivity, $y'_0 = \varphi_s(y_0)$ for some $s \in [-1/4, 1/4]$. In particular $y' = \varphi_{s+s_0-s'_0}(y)$ with $|s + s_0 - s'_0| \leq 3/4$.

Using the Local injectivity, there exists $\tau \in [-1/4, 1/4]$ such that $\varphi_{t+\tau}(y)$ is $r/2$ -close to x and satisfies $\pi_x(\varphi_{t+\tau}(y)) = \pi_x(\varphi_t(y))$. Since $y_0 := \varphi_{s_0}(y)$ and $\varphi_{t+\tau}(y)$ are both $r/2$ -close to x , the “No small period” assumption implies that $|t + \tau - s_0|$ is either larger or equal to 2, or smaller than $1/2$. Since by definition $t \geq 1$ one has $t + \tau - s_0 \geq 2$.

Thus, $\varphi_t(y)$ is the image of y' at the time $t - (s_0 + s + s'_0) = (t + \tau - s_0) - (s'_0 + \tau + s_0)$, which is larger than $2 - 3/4 > 1$. By minimality in the definition of t , $\varphi_t(y)$ is a forward iterate of $\varphi_{t'}(y')$. In a similar way $\varphi_{t'}(y')$ is a forward iterate of $\varphi_t(y)$. Hence these two points coincide. \square

The next lemma shows that the orbits under f correspond to (the projection by π_x of) orbits under φ restricted to \widehat{R} .

Lemma 6.11. *For any $y \in \widehat{R}$, let $t = \min\{s \geq 1, d(\varphi_s(y), x) \leq r_0/3 \text{ and } \varphi_s(y) \in \widehat{R}\}$. Then for any $s \in [0, t]$ such that $\varphi_s(y) \in \widehat{R}$, we have $s \notin (3/4, 3/2)$. Moreover:*

- if $s \leq 3/4$, $\pi_x(\varphi_s(y)) = \pi_x(y)$,
- if $s \geq 3/2$, $\pi_x(\varphi_s(y)) = \pi_x(\varphi_t(y))$ (which coincides with $f(\pi_x(y))$) and $s \geq t - 1/4$.

Proof. The proof of Lemma 6.10 showed that if $y, y' \in \widehat{R}$ have the same projection by π_x , then $y' = \varphi_s(y)$ for some $|s| \leq 3/4$.

On the other hand if $y, y' \in \widehat{R}$ belong to the same orbit (i.e. $y' = \varphi_s(y)$) but have different projection by π_x , then $|s| > 3/2$. Indeed, by the Local injectivity, there exists $y_0 = \varphi_{s_0}(y)$ and $y'_0 = \varphi_{s'_0}(y')$ which are $r/2$ -close to x such that $|s_0| + |s'_0| \leq 1/2$. Since y_0, y'_0 have different projections by π_x , the Local invariance implies that $|s - s_0 + s'_0| \geq 2$. This gives $|s| > 3/2$.

These two properties imply that for any $y, y' \in \widehat{R}$ with $y' = \varphi_s(y)$, then $s \notin (3/4, 3/2)$ and these points have the same projection by π_x if $|s| \leq 3/4$.

Let us assume that $3/2 \leq s \leq t$. One considers by the Local injectivity s' such that $|s - s'| \leq 1/4$, $\varphi_{s'}(y)$ is $r/2$ -close to x and $\pi_x(\varphi_{s'}(y)) = \pi_x(\varphi_s(y))$. Then $s' \geq 1$ and by definition of t , one gets $s' \geq t$. Consequently $s \geq t - 1/4$ and $\pi_x(\varphi_s(y)) = \pi_x(\varphi_t(y))$. \square

Lemma 6.12. *The map f is continuous.*

Proof. Fix $u \in \Lambda$. There is $y \in \widehat{R}$ such that $d(x, y) < r/2$ and $u = \pi_x(y)$. For any $u' \in \Lambda$ close to u , there exists $y' \in \widehat{R}$ with the same properties and such that $\pi_y(y')$ is arbitrarily close to 0_y . So by the Local injectivity and the “No small period” assumption, one can choose y' arbitrarily close to y . Let t, t' be the times associated to y, y' as in Lemma 6.11.

By continuity of the flow, $\varphi_t(y')$ is r_0 -close to x and has a projection by π_x close to $\pi_x(\varphi_t(y)) \in \Lambda$. Since Λ is compact and contained in the interior of R , $\pi_x(\varphi_t(y'))$ belongs to R (hence to Λ). We claim that it coincides with $f(u')$ which will conclude the proof.

Let us assume by contradiction that $\pi_x(\varphi_t(y')) \neq \pi_x(\varphi_{t'}(y'))$. Lemma 6.11 implies that $t \geq t' + 3/2$. Then $\varphi_{t'}(y)$ is $r_0/2$ -close to x and projects by π_x to R . We get $\varphi_{t'}(y) \in \widehat{R}$ with $1 \leq t' \leq t - 3/2$, contradicting Lemma 6.11. \square

By repeating the above construction for negative times, we will obtain another map. Then Lemma 6.11 shows that it is the inverse of f . Since φ is minimal, this gives:

Corollary 6.13. *f is a homeomorphism and induces a minimal dynamics on K .*

Extension of f as a local diffeomorphism. For any $u \in \Lambda$ we choose y and t as in the definition of f . One gets a local C^2 -diffeomorphism f_u from a (uniform) neighborhood of u to a (uniform) neighborhood of $f(u)$ defined by $\pi_x P_t \pi_y$. (The uniformity comes from the fact that t is uniformly bounded. The Local invariance shows it does not depend on y .)

Lemma 6.14. *There exists a local diffeomorphism on a neighborhood of Λ which extends f and each f_u .*

Proof. For $u', u \in \Lambda$ that are close, we have to show that the diffeomorphisms $f_u, f_{u'}$ matches on uniform neighborhood of u and u' . Let us consider y, y' and t, t' defining the local diffeomorphisms, such that y, y' are $r/2$ -close to x . Note that from the proof of Lemma 6.12, y, y' (resp. t, t') can be chosen arbitrarily close if u, u' are close.

Take $z \in \mathcal{N}_x$ in the intersection of the domains close to $\pi_x(u)$ and $\pi_x(u')$. Its projection by π_y and $\pi_{y'}$ gives $v \in \mathcal{N}_y$ and $v' \in \mathcal{N}_{y'}$ close to 0_y and $0_{y'}$ whose orbits under P remain close to the zero section during the time t (resp. t'). By Global invariance, there exists an increasing homeomorphism θ of \mathbb{R} close to the identity such that

$$f_u(z) = \pi_x P_t(v) = \pi_x P_{\theta(t)}(v') = \pi_x P_{t'}(v') = f_{u'}(z).$$

One deduces that the maps f_u define a C^2 map on a neighborhood of Λ which extends f . Since the same construction can be applied with the local diffeomorphisms f_u^{-1} , one concludes that f is a diffeomorphism. \square

Extensions of the bundles E, F .

Lemma 6.15. *The tangent bundle over Λ admits a splitting $E \oplus F$ which is invariant and dominated by f . Moreover E is uniformly contracted by f on Λ if and only if \mathcal{E} is uniformly contracted by the flow (P_t) on K .*

Proof. At each point $u \in \Lambda$ we define the spaces $E(u), F(u)$ as the image by $D\pi_x(0_y)$ of $\mathcal{E}(y), \mathcal{F}(y)$ where $y \in \widehat{R}$ and $\pi_x(y) = u$. These spaces are well defined: if $y' \in \widehat{R}$ also satisfies $\pi_x(y') = u$, then $y' = \varphi_t(y)$ for some $t \in [-1, 1]$; the Local injectivity and the invariance of the bundles \mathcal{E} implies that $D\pi_x(0_y) \cdot \mathcal{E}(y) = D\pi_x(0_{y'}) \cdot \mathcal{E}(y')$. The same holds for \mathcal{F} . The continuity of the families $\pi_{y,x}$ and of the bundles \mathcal{E}, \mathcal{F} over the 0-section of \mathcal{N} implies that E, F are continuous over Λ .

Let us consider $u' = f(u)$ and two points $y, y' \in \widehat{R}$ that are $r_0/2$ -close to x such that $\pi_x(y) = u$ and $\pi_x(y') = u'$. Then, there exists $t > 0$ such that $\varphi_t(y) = y'$ so that $DP_t(0_y) \cdot \mathcal{E}(y) = \mathcal{E}(y')$. Consequently, we obtain the invariance of E by Df :

$$Df(u) \cdot E(u) = D\pi_x(0_{y'}) \circ DP_t(0_y) \circ D\pi_y(0_x) \circ D\pi_x(0_y) \cdot \mathcal{E}(y) = D\pi_x(0_{y'}) \cdot \mathcal{E}(y') = E(u').$$

Note that the splitting $E \oplus F$ on Λ is dominated for the dynamics of Df since $Df^N(u)$ coincides for N large with $D\pi_x \circ DP_t \circ D\pi_y$ for some large $t > 0$ and some $y \in \widehat{R}$ satisfying $u = \pi_x(y)$ and since $\mathcal{E} \oplus \mathcal{F}$ is dominated for the dynamics of DP_t . Since all orbits of f correspond to the orbit under φ (by minimality of K), the argument proves that E is uniformly contracted by f on Λ if and only if \mathcal{E} is uniformly contracted by the flow (P_t) on K . \square

End of the proof of Proposition 6.1 in the minimal case. Since the set K is minimal and not a periodic orbit, the set Λ does not contain any periodic orbit. Note that Λ cannot contain a closed curve tangent to E nor a closed curve tangent to F since Λ is contained in R which has arbitrarily small diameter. So Pujals-Sambarino's theorem applies and Λ is a hyperbolic set for f . This implies that E and \mathcal{E} are uniformly contracted by f and (P_t) respectively as required.

The proof of Proposition 6.1 is now complete. \square

6.3 Fibered version of Mañé-Pujals-Sambarino's theorem

Proof of Theorem C. Let us assume that a local fibered flow (\mathcal{N}, P) satisfies the assumptions of Theorem C. We suppose furthermore that K does not contain a normally expanded irrational torus, and that \mathcal{E} is uniformly contracted over each periodic orbit.

Assume by contradiction that the bundle \mathcal{E} is not uniformly contracted. Then, there exists a non-empty invariant compact subset $\tilde{K} \subset K$ such that

- \mathcal{E} is not uniformly contracted over \tilde{K} ,
- but \mathcal{E} is uniformly contracted over any invariant compact proper subset $\tilde{K}' \subset \tilde{K}$.

The assumptions (A1), (A2), (A3) are satisfied and the Theorem 4.1 can be applied to \tilde{K} . By our assumptions, the two first conclusions are not satisfied, hence the bundle \mathcal{E} over \tilde{K} is topologically contracted. Note also that since \mathcal{E} is contracted over periodic orbits of K , the set \tilde{K} is not reduced to a periodic orbit. The properties (B1), (B2), (B3), (B4) are satisfied on \tilde{K} .

Since \mathcal{E} is not uniformly contracted over \tilde{K} and is one-dimensional, there exists an ergodic measure μ with support contained in \tilde{K} whose Lyapunov exponent along \mathcal{E} non-negative. By domination, the Lyapunov exponent along \mathcal{F} is positive. Since \mathcal{E} is uniformly contracted over any invariant proper compact subset, the support of μ coincides with \tilde{K} . Proposition 6.1 applies to \tilde{K} and μ and contradicts the fact that the Lyapunov exponent of μ along \mathcal{E} is non-negative. Hence \mathcal{E} is uniformly contracted over K . \square

7 Generalized Mañé-Pujals-Sambarino theorem for singular flows

In this section, we will prove Theorem A' by using Theorem B and Theorem C. We consider a manifold M and an invariant compact set Λ for a C^2 vector field X on M whose singularities are hyperbolic and have simple real eigenvalues (in particular their number is finite). The results trivially holds for isolated singularities (since by assumption they admit a negative Lyapunov exponent). Hence, it is enough to assume that the set of regular orbits is dense in Λ .

In the last subsection we will prove the easy side of Theorem A'. In all the other subsections, we assume that the linear Poincaré flow on $\Lambda \setminus \text{Sing}(X)$ has a dominated splitting $\mathcal{N} = \mathcal{E} \oplus \mathcal{F}$ and prove the existence of a dominated splitting for the tangent flow.

7.1 Compactification

One first applies Theorem B and gets maps

$$i: \Lambda \setminus \text{Sing}(X) \rightarrow K := \widehat{\Lambda} \text{ and } I: \mathcal{N}M|_{\Lambda \setminus \text{Sing}(X)} \rightarrow \mathcal{N} := \widehat{\mathcal{N}M}.$$

The set K is the closure of $\Lambda \setminus \text{Sing}(X)$ in the blowup \widehat{M} of M at each singularity $\text{Sing}(X) \cap \Lambda$, so that the map i is the canonical injection of $\mathcal{N}M|_{\Lambda \setminus \text{Sing}(X)}$ in \mathcal{N} , and I is the canonical injection of $\mathcal{N}M|_{\Lambda \setminus \text{Sing}(X)}$ inside the compactification \mathcal{N} . In the following we drop the injections i and I .

The set K is endowed with a flow $\widehat{\varphi}$ which extends the flow φ on $\Lambda \setminus \text{Sing}(X)$. The rescaled sectional Poincaré flow extends as a C^2 local fibered flow \widehat{P}^* in a neighborhood of the 0-section of \mathcal{N} over K . The fibers of \mathcal{N} have dimension $\dim(M) - 1$.

7.2 Identifications

We now choose an open set $U \subset K$ such that $K \setminus U$ is an arbitrarily small neighborhood of the compact set $K \setminus (\Lambda \setminus \text{Sing}(X))$. For any singularity $\sigma \in \Lambda$, we denote by d^s, d^u its stable and unstable dimensions. Since it is hyperbolic, $d^s + d^u = \dim M$. Let us choose a C^1 chart at σ which identifies σ with $0 \in \mathbb{R}^{d^s+d^u}$, and the local stable and unstable manifolds with $\mathbb{R}^{d^s} \times \{0\}$ and $\{0\} \times \mathbb{R}^{d^u}$. There exists two (closed) differentiable balls $B^s \subset \mathbb{R}^{d^s} \times \{0\}$ and $B^u \subset \{0\} \times \mathbb{R}^{d^u}$ that are transverse to the linear vector field $u \mapsto DX(0).u$ on $\mathbb{R}^{d^s+d^u} \setminus \{0\}$. For $\varepsilon > 0$ small, the vector field X is transverse to the boundary of the ε -scaled neighborhood $B_\sigma = \varepsilon.(B^s \times B^u)$ at any point of $\partial B^s \times \text{Interior}(B^u)$ and of $\text{Interior}(B^s) \times \partial B^u$. Note that for $x \in \partial B^s \times \partial B^u$, the image $\varphi_t(x)$ does not belong to B_σ for any small $t \neq 0$. The union of the B_σ s for all singularities is a neighborhood of $K \setminus (\Lambda \setminus \text{Sing}(X))$ and its complement defines the open set U . Thus, the “Transverse boundary” property holds.

Since there is no fixed point of $\widehat{\varphi}$ in \overline{U} , one can rescale the time (i.e. consider the new flow $t \mapsto \widehat{\varphi}_{t/C}$ for some large $C > 1$) so that any periodic orbit which meets \overline{U} has period larger than 10. Then the “No small period” property follows from the continuity of the flow.

If $\overline{\beta} > 0$ is small enough, for any point x in a neighborhood of \overline{U} the image of $B(0, \overline{\beta}) \subset \mathcal{N}_x$ by the exponential map \exp_x is transverse to the vector field X . Consequently, for $\varepsilon > 0$ small, if $\beta_0 \in (0, \overline{\beta})$ and $r_0 > 0$ are much smaller than ε , for any point $y \in M$ such that $d(x, y) < r_0$ and any $u \in B(0, \beta_0) \subset \mathcal{N}_y$, there exists a unique $s \in (-\varepsilon, \varepsilon)$ satisfying $\varphi_s(\exp_y(u))$ belongs to $\exp_x(B(0, \overline{\beta}))$. After rescaling, we thus define the identification

$$\pi_{y,x}(u) := \|X(x)\|^{-1} \cdot \exp_x^{-1} \circ \varphi_s \circ \exp_y(\|X(y)\|.u).$$

Since X is C^2 , the map $\pi_{y,x}$ is C^2 also. By the uniqueness of the parameter s , we obtain the relation $\pi_{z,x} \circ \pi_{y,z} = \pi_{y,x}$.

The Local injectivity now follows immediately by choosing $t = s$ as in the definition of the identification $\pi_{y,x}$. Let us consider $x, y \in U$, $t \in [-2, 2]$ and $u \in B(0_y, \beta_0)$ such that y and $\varphi_t(y)$ are r_0 -close to x . If r_0 has been chosen small enough the “No small period” property implies that t is small. Then the uniqueness of s in the definition of the identification π implies $\pi_x \circ \widehat{P}_t^*(u) = \pi_x(u)$. This gives the Local invariance.

7.3 Global invariance

The following two lemmas follow from the fact that the vector field X is almost constant in the $\varkappa\|X(y)\|$ -neighborhood of y for $\varkappa > 0$ small enough,

Lemma 7.1. *For any $\rho > 0$, there exists $\delta > 0$ with the following property.*

If $y \in \Lambda \setminus \text{Sing}(X)$ and if $z = \exp_y(u)$ for some $u \in B(0_y, \delta\|X(y)\|) \subset \mathcal{N}_y$, then for any $s \in (0, 1)$, there exists a unique $s' \in (0, 2)$ such that $\varphi_{s'}(z) = \exp_{\varphi_s(y)} \circ P_s(u)$. Moreover $\max(s/s', s'/s) < 1 + \rho/3$.

Lemma 7.2. *For any $\delta, t_0 > 0$, there exists $\beta > 0$ with the following property.*

For any $y \in \Lambda \setminus \text{Sing}(X)$ and $z \in M$ such that $d(z, y) \leq 10\beta\|X(y)\|$, there exists a unique $t \in (-t_0, t_0)$ such that $\varphi_t(z)$ belongs to the image by \exp_y of $B(0_y, \delta\|X(y)\|) \subset \mathcal{N}_y$.

We can now check the last item of the Definition 3.4 for the local fibered flow \widehat{P}^* . Let us fix $\delta, \rho > 0$ small: by reducing δ , one can assume that Lemma 7.1 above holds. One then chooses $t_0 > 0$ small such that $d(x, \varphi_t(x)) < \delta/3$ for any $x \in M$ and $t \in [-t_0, t_0]$. One fixes $r > 0$ and $\beta \in (0, \delta)$ small such that:

- a- for any y, y' and $u \in \mathcal{N}_y$, $u' \in \mathcal{N}_{y'}$ as in the statement of Global invariance, then $\varphi_t \exp_y(\|X(y)\|.u) = \exp_{y'}(\|X(y')\|.u')$ for some $t \in [-t_0, t_0]$ (arguing as in the proof of Local injectivity),
- b- δ, β satisfy the Lemma 7.2,
- c- for any $x \in M$, $d(x, \exp_x(w)) < \delta/3$ for any $w \in T_x M$ satisfying $\|w\| \leq 10\beta\|X(x)\|$.

Consider any y, y' , $u \in \mathcal{N}_y$, $u' \in \mathcal{N}_{y'}$ and I, I' as in the statement of the Global invariance. Lemma 7.1 can be applied to y and $z = \exp_y(\|X(y)\|.u)$: for each $s \in (0, 1)$, one defines $\theta_0(s) \in (0, 2)$ to be equal to the s' given by Lemma 7.1. The map θ_0 is $(1 + \rho/3)$ -bi-Lipschitz and increasing. Moreover $\theta_0(0) = 0$. Since $\|\widehat{P}_s^*(u)\| \leq \beta < \delta$ for any $s \in I$, one has $P_s(\|X(y)\|.u) \in B(0, \delta\|X(\varphi_s(y))\|)$ and one can apply inductively Lemma 7.1 to the points $\varphi_s(y)$ and $P_s(\|X(y)\|.u)$, which defines θ_0 on I . One gets:

$$\forall s \in I, \quad \exp_{\varphi_s(y)} \circ P_s(\|X(y)\|.u) = \varphi_{\theta_0(s)}(z).$$

The same argument for y' and $z' = \exp_{y'}(\|X(y')\|.u')$ defines a map $\theta'_0: I' \rightarrow \mathbb{R}$.

Let us now consider $s \in I \cap \theta_0^{-1} \circ \theta'_0(I')$. By (a), since $\pi_y(u') = u$, there exists $t \in [-t_0, t_0]$ such that $\varphi_t(z) = z'$. By the definition of θ_0 , the points $\varphi_s(y)$ and $\varphi_{\theta_0(s)}(z)$ are $\delta/3$ -close. Since $|t| \leq t_0$, the points $\varphi_{\theta_0(s)}(z)$ and $\varphi_{\theta_0(s)+t}(z) = \varphi_{\theta_0(s)}(z')$ are $\delta/3$ -close. Since $\theta_0(s) \in \theta'_0(I')$ and using (c) above, the points $\varphi_{\theta_0(s)}(z')$ and $\varphi_{(\theta'_0)^{-1} \circ \theta_0(s)}(y')$ are $\delta/3$ -close. Consequently, the points $\varphi_s(y)$ and $\varphi_{\theta(s)}(y')$ are δ -close, where $\theta = (\theta'_0)^{-1} \circ \theta_0$. Note that θ is bi-Lipschitz for the constant $(1 + \rho/3)^2 < 1 + \rho$ and satisfies $\theta(0) = 0$. This proves the first part of the Global invariance.

Finally we take $v \in \mathcal{N}_y$, $v' = \pi_{y'}(v)$ in $\mathcal{N}_{y'}$ such that $\|\widehat{P}_s^*(v)\| < \beta$ for each $s \in I \cap \theta^{-1}(I')$. Set $\zeta = \exp_y(\|X(y)\|.v)$. By Lemma 7.2, there exists a unique $t' \in (-t_0, t_0)$ such that $\varphi_{t'}(\zeta) = \exp_{y'}(w')$ for some $w' \in B(0, \delta) \subset \mathcal{N}_{y'}$. By definition of $\pi_{y,y'}$, it coincides with $\exp_{y'}(\|X(y')\|.v')$.

Arguing as above, there exists θ_1 such that $\exp_{\varphi_s(y)} \circ P_s(\|X(y)\|.v) = \varphi_{\theta_1(s)}(\zeta)$ and $\theta_1(0) = 0$. $d(\varphi_{\theta_1(s)}(\zeta), \varphi_s(y))$ is smaller than $\|P_s(\|X(y)\|.v)\|$ and $\beta\|X(\varphi_s(y))\|$. Similarly,

$$d(\varphi_{\theta_0(s)}(z), \varphi_s(y)) \leq \beta\|X(\varphi_s(y))\|.$$

$$d(\varphi_{\theta'_0(s')}(z'), \varphi_{s'}(y')) \leq \beta\|X(\varphi_{s'}(y'))\|.$$

Since $z' = \varphi_t(z)$, to each $s \in I \cap \theta^{-1}(I')$ one associates s' such that $\theta_0(s) = \theta'_0(s') + t$, one gets

$$d(\varphi_{\theta_0(s)}(z), \varphi_{s'}(y')) \leq \beta\|X(\varphi_{s'}(y'))\|.$$

If β has been chosen small enough, one deduces that $\beta\|X(\varphi_s(y))\|$ and $\beta\|X(\varphi_{\theta_0(s)}(z))\|$ are smaller than $2\beta\|X(\varphi_{s'}(y'))\|$. Hence,

$$d(\varphi_{\theta_1(s)}(\zeta), \varphi_{s'}(y')) \leq 5\beta\|X(\varphi_{s'}(y'))\|. \quad (15)$$

By Lemma 7.2, one can find $\sigma(s) \in (-t_0, t_0)$ such that $\varphi_{\theta_1(s)+\sigma(s)}(\zeta)$ belongs to the image by $\exp_{\varphi_{s'}(y')}$ of $B(0, \delta) \subset \mathcal{N}_{\varphi_{s'}(y')}$. In the case $s' = 0$, since t and σ are small, the definition of π_y gives $\varphi_{\theta_1(s)+\sigma(s)}(\zeta) = \exp_{y'}(\|X(y')\|.v')$. Applying Lemma 7.1 inductively, one has that $\varphi_{\theta_1(s)+\sigma(s)}(\zeta) = \exp_{\varphi_{s'}(y')}(P_{s'}(\|X(y')\|.v'))$ for any $s \in I \cap \theta^{-1}(I')$. By (15) and Lemma 7.2, one deduces $\|P_{s'}(\|X(y')\|.v')\| \leq \delta\|X(\varphi_{s'}(y'))\|$, that is $\|\widehat{P}^*(v')\| \leq \delta$ as wanted.

We have obtained

$$\varphi_{\sigma(s)} \circ \exp_{\varphi_s(y)} \circ P_s(\|X(y)\|.v) = \exp_{\varphi_{s'}(y')}(P_{s'}(\|X(y')\|.v')).$$

When $\varphi_s(y)$ and $\varphi_{s'}(y')$ are in a neighborhood of \overline{U} , one deduces by definition of identification,

$$\pi_{\varphi_s(y)} \circ P_{s'}(\|X(y')\|.v') = P_s(\|X(y)\|.v).$$

By definition of θ and s' , one notices that $\theta(s)$ and s' are close. Hence

$$\pi_{\varphi_{s'}(y')} \circ P_{\theta(s)}(\|X(y')\|.v') = P_{s'}(\|X(y')\|.v').$$

This gives $\pi_{\varphi_s(y)} \circ \widehat{P}_{\theta(s)}^*(v') = \widehat{P}_s^*(v)$ and completes the proof of the Global invariance.

7.4 Dominated splitting

We have assumed that the linear Poincaré flow ψ on $\Lambda \setminus \text{Sing}(X)$ admits a dominated splitting, denoted by $\mathcal{NM}|_{\Lambda \setminus \text{Sing}(X)} = \mathcal{E} \oplus \mathcal{F}$. It extends as a dominated splitting $\mathcal{N} = \widehat{\mathcal{E}} \oplus \widehat{\mathcal{F}}$ over K for the extended rescaled linear Poincaré flow $\widehat{\psi}^*$ (hence for \widehat{P}^*). Indeed:

- dominated splittings are invariant under rescaling, hence $\mathcal{E} \oplus \mathcal{F}$ is a dominated splitting for ψ^* (and $\widehat{\psi}^*$) over $\Lambda \setminus \text{Sing}(X)$;
- for continuous linear cocycles, dominated splittings extend to the closure.

The existence of a dominated splitting for the tangent flow on Λ can be restated as the uniform contraction of the bundle $\widehat{\mathcal{E}}$.

Proposition 7.3. *Under the previous assumptions, these two properties are equivalent:*

- I– *There exists a dominated splitting $T_\Lambda M = E \oplus F$ for the tangent flow $D\varphi$ with $\dim(E) = \dim(\widehat{\mathcal{E}})$ and $X \subset F$;*
- II– *$\widehat{\mathcal{E}}$ is uniformly contracted by \widehat{P}^* (and $\widehat{\psi}^*$) over K .*

Proof. Let us prove $I \Rightarrow II$. From Property I, we have a dominated splitting between E and $\mathbb{R}X$, hence there exists $C > 0$ and $\lambda \in (0, 1)$ such that for any $t > 0$ and any $x \in \Lambda$,

$$\|D\varphi_t|E(x)\| \leq C\lambda^t \|D\varphi_t|\mathbb{R}X(x)\| = C\lambda^t \frac{\|X(\varphi_t(x))\|}{\|X(x)\|}.$$

Since the angle between E and X is uniformly bounded away from zero, the projection of $E(z)$ on $X(z)^\perp$ and its inverse are uniformly bounded, hence the ratio between $\|D\varphi_t|E(x)\|$ and $\|\psi_t|\widehat{\mathcal{E}}(x)\|$ is bounded. This implies that there exists C' such that:

$$\|\widehat{\psi}_t^*|\widehat{\mathcal{E}}(x)\| = \frac{\|X(x)\|}{\|X(\varphi_t(x))\|} \|\psi_t|\widehat{\mathcal{E}}(x)\| \leq C'\lambda^t.$$

The bundle $\widehat{\mathcal{E}}$ is thus uniformly contracted over $\Lambda \setminus \text{Sing}(X)$, hence over K also. This gives Property II. The implication $II \Rightarrow I$ is a restatement of [GY, Lemma 2.13]. \square

7.5 Uniform contraction of $\widehat{\mathcal{E}}$ near the singular set

From the assumptions of Theorem A', any singularity $\sigma \in \Lambda$ has a dominated splitting $T_\sigma M = E^{ss} \oplus F$, where E^{ss} is uniformly contracted, has the same dimension as \mathcal{E} , and the associated invariant manifold $W^{ss}(\sigma)$ intersects Λ only at σ . Let V be a small open neighborhood of the compact set $K \setminus (\Lambda \setminus \text{Sing}(X))$.

Lemma 7.4. *The bundle $\widehat{\mathcal{E}}$ is uniformly contracted on V by \widehat{P}^* .*

Proof. We use the notations and the discussions of section 2.4. For each singularity $\sigma \in \Lambda$, let Δ_σ be the set of unit vectors $u \in E^{cu}(\sigma) \subset T_\sigma M$. It is compact and $\widehat{\varphi}$ -invariant. The splitting at σ induces a dominated splitting $E^{ss} \oplus E^{cu}$ of the extended bundle \widehat{TM} over Δ_σ .

For regular orbits near Δ_σ , the lines $\mathbb{R}X$ are close to $E^{cu}(\sigma)$, hence have a uniform angle with E^{ss} . Consequently, the dominated splitting $E^{ss} \oplus E^{cu}$ over Δ_σ projects on the extended normal bundle \widehat{N} over Δ_σ as a splitting $\mathcal{E}' \oplus \mathcal{F}'$ where $\dim(\mathcal{E}') = \dim(E^{ss}) = 1$, which is dominated for the linear Poincaré flow ψ (and hence for ψ^* also).

The dominated splitting $E^{ss} \oplus E^{cu}$ induces a dominated splitting between E^{ss} and the extended line field $\widehat{\mathbb{R}X}_1$. The proof of Proposition 7.3 above shows that the extended rescaled linear Poincaré flow $\widehat{\psi}^*$ contracts \mathcal{E}' (above Δ_σ).

On $K \cap \Delta_\sigma$, the dominated splittings $\widehat{\mathcal{E}} \oplus \widehat{\mathcal{F}}$ and $\mathcal{E}' \oplus \mathcal{F}'$ have the same dimensions, hence coincide. Moreover since $W^{ss}(\sigma) \cap \Lambda = \{\sigma\}$, we have $p^{-1}(\sigma) \cap K = \Delta_\sigma \cap K$, where p denotes the projection $K \rightarrow \Lambda$. This shows that $\widehat{\mathcal{E}}$ is uniformly contracted by \widehat{P}^* over $p^{-1}(\text{Sing}(X) \cap \Lambda) \cap K$, hence on any small neighborhood V . \square

7.6 Periodic orbits and normally expanded invariant tori

We now check that the two first conclusions of Theorem C do not hold.

Lemma 7.5. *For any periodic orbit \mathcal{O} in K , the bundle $\widehat{\mathcal{E}}|_{\mathcal{O}}$ is uniformly contracted.*

Proof. By Lemma 7.4, the bundle $\widehat{\mathcal{E}}$ is contracted over periodic orbits contained in V . The other periodic orbits are lifts (by the projection $p: \widehat{M} \rightarrow M$) of orbits in $\Lambda \setminus \text{Sing}(X)$. From the assumptions of Theorem A', the Lyapunov exponents along \mathcal{E} are all negative for periodic orbits in $\Lambda \setminus \text{Sing}(X)$. This concludes. \square

Lemma 7.6. *There do not exist any normally expanded irrational torus \mathcal{T} for $(K, \widehat{\varphi})$.*

Proof. We now use the fact that M is three-dimensional. Let us assume by contradiction that there exists a normally expanded irrational torus \mathcal{T} for $(K, \widehat{\varphi})$. By Lemma 3.15, the bundle $\widehat{\mathcal{F}}$ is uniformly expanded over \mathcal{T} . Since it does not contain fixed point of $\widehat{\varphi}$, it projects by p in $\Lambda \setminus \text{Sing}(X)$. By construction, the dynamics of $\widehat{\psi}^*$ over \mathcal{T} and ψ^* over $p(\mathcal{T})$ are the same, hence \mathcal{F} is uniformly expanded over $p(\mathcal{T})$ by ψ^* . Since $p(\mathcal{T}) \cap \text{Sing}(X) = \emptyset$, one deduces that \mathcal{F} is uniformly expanded over $p(\mathcal{T})$ by ψ . Proposition 7.3 implies that the tangent flow over $p(\mathcal{T})$ has a dominated splitting $TM|_{p(\mathcal{T})} = E^c \oplus E^{uu}$, with $\dim(E^{uu}) = 1$.

As a partially hyperbolic set each $x \in p(\mathcal{T})$ has a strong unstable manifold $W^{uu}(x)$. Note that $W^{uu}(x) \cap p(\mathcal{T}) = \{x\}$ (since the dynamics is topologically equivalent to an irrational flow). Then [BC2] implies that $p(\mathcal{T})$ is contained in a two-dimensional submanifold Σ transverse to E^{uu} and locally invariant by φ_1 : there exists a neighborhood U of $p(\mathcal{T})$ in Σ such that $\varphi_1(U) \subset \Sigma$. Since $p(\mathcal{T})$ is homeomorphic to \mathbb{T}^2 , it is open and closed in Σ , hence coincides with Σ . This shows that $p(\mathcal{T})$ is C^1 -diffeomorphic to \mathbb{T}^2 , normally expanded and carries a dynamics topologically equivalent to an irrational flow. This contradicts the assumptions of Theorem A'. Consequently there do not exist any normally expanded irrational torus \mathcal{T} for $(K, \widehat{\varphi})$. \square

7.7 Proof of the domination of the tangent flow

Under the assumptions of Theorem A', the fibers of \mathcal{E} and \mathcal{F} are one-dimensional. Note that one can choose U after V such that $U \cup V = K$. We have thus shown that \widehat{P}^* over K satisfies the setting of Theorem C, that $\widehat{\mathcal{E}}$ is uniformly contracted over the periodic orbits and that there is no normally expanded irrational torus. One deduces that $\widehat{\mathcal{E}}$ is uniformly contracted by \widehat{P}^* above K . Proposition 7.3 then implies that there exists a dominated splitting $T_\Lambda M = E \oplus F$ such that E is one-dimensional (as for $\widehat{\mathcal{E}}$) and $X(x) \subset F(x)$ for any $x \in \Lambda$.

This shows one side of Theorem A': a domination of the linear Poincaré flow implies a domination $TM|_\Lambda = E \oplus F$ of the tangent flow with $\dim(E) = 1$.

7.8 Proof of the domination of the linear Poincaré flow

The other direction of Theorem A' is easier.

Proposition 7.7. *Under the assumptions of Theorem A', if there exists a dominated splitting $TM|_\Lambda = E \oplus F$ with $\dim(E) = 1$ for the tangent flow $D\varphi$, then:*

- $X(x) \subset F(x)$ for any $x \in \Lambda$,
- the linear Poincaré flow on $\Lambda \setminus \text{Sing}(X)$ is dominated,
- E is uniformly contracted.

Proof. We first prove that $X(x) \in F(x)$ for any $x \in \Lambda$. Otherwise, using the domination, there exists a non-empty invariant compact subset Λ' such that $X(x) \in E(x)$ for any $x \in \Lambda'$ and there exists a regular orbit in Λ which accumulates in the past on Λ' .

Let μ be an ergodic measure on Λ' . If $\text{supp}(\mu)$ is not a singularity, the domination implies that all the Lyapunov exponents along \mathcal{F} are positive, hence the measure is hyperbolic and is supported on a periodic orbit that is a source. This contradicts the assumptions of Theorem A'.

If μ is supported on a singularity σ , it is by construction limit of regular points $x_n \in \Lambda$ such that $\mathbb{R}X(x_n)$ converges towards $E(\sigma)$. This implies that one of the separatrices of $W^{ss}(\sigma)$ is contained in Λ , a contradiction. The first item follows.

Since $X \subset F$ and the angle between E and F is bounded away from zero, the projection of $T_x M = E \oplus F$ to \mathcal{N}_x along $\mathbb{R}X(x)$ defines a splitting $\mathcal{E}(x) \oplus \mathcal{F}(x)$ into one-dimensional subspaces, for each $x \in \Lambda \setminus \text{Sing}(X)$. This splitting is continuous and invariant under ψ .

Let us consider two non-zero vectors $u \in \mathcal{E}(x)$ and $v \in \mathcal{F}(x)$ and $t > 0$. We have

$$\|\psi_{-t}.v\| \leq \|D\varphi_{-t}.v\|.$$

Since E and \mathcal{E} are uniformly transverse to $\mathbb{R}X$, there is a constant $C > 0$ such that

$$\|\psi_{-t}.u\| \geq C^{-1}\|D\varphi_{-t}.u\|.$$

The domination $E \oplus F$ thus implies the domination $\mathcal{E} \oplus \mathcal{F}$.

One can then apply Lemma 7.8, whose proof is contained in the proof of [BGY, Lemma 3.6].

Lemma 7.8. *Consider a C^1 vector field and an invariant compact set Λ endowed with a dominated splitting $T_\Lambda M = E \oplus F$ such that E is one-dimensional, $X(x) \in F(x)$ for any $x \in \Lambda$ and $E(\sigma)$ is contracted for each $\sigma \in \Lambda \cap \text{Sing}(X)$. Then E is uniformly contracted.*

The bundle E is thus uniformly contracted. This ends the proof of Proposition 7.7 □

The proof of Theorem A' is now complete. □

8 C^1 -generic three-dimensional vector fields

In this section we prove Theorem A, Corollary 1.1 and the Main Theorem.

8.1 Chain-recurrence and genericity

For $\varepsilon > 0$, we say that a sequence x_0, \dots, x_n is an ε -pseudo-orbit if for each $i = 0, \dots, n-1$ there exists $t \geq 1$ such that $d(\varphi_t(x_i), x_{i+1}) < \varepsilon$. A non-empty invariant compact set K for a flow φ is *chain-transitive* if for any $x, y \in K$ (possibly equal) and any $\varepsilon > 0$, there exists a ε -pseudo-orbit $x_0 = x, \dots, x_n = y$ with $n \geq 1$.

A *chain-recurrence class* of φ is a chain-transitive set which is maximal for the inclusion. The *chain-recurrent set* of φ is the union of the chain-recurrence classes. See [Co].

We then recall known results on generic vector fields. We say that a property is satisfied by generic vector fields in $\mathcal{X}^r(M)$ if it holds on a dense G_δ subset of $\mathcal{X}^r(M)$.

Theorem 8.1. *For any manifold M , any $r \geq 1$ and any generic vector field X in $\mathcal{X}^r(M)$,*

- *each periodic orbit or singularity is hyperbolic and has simple (maybe complex) eigenvalues,*
- *there do not exist any invariant subset \mathcal{T} which is diffeomorphic to \mathbb{T}^2 , normally expanded and supports a dynamics topologically equivalent to an irrational flow.*

Proof. The first part is similar to the proof of the Kupka-Smale property [K, S].

The second part can be obtained from a Baire argument by showing that if $X \in \mathcal{X}^r(M)$ preserves an invariant subset \mathcal{T} which is diffeomorphic to \mathbb{T}^2 , normally expanded, and which supports a dynamics topologically equivalent to an irrational flow, then there exists a neighborhood U of \mathcal{T} and an open set of vector fields X' C^r -close to X whose maximal invariant set in U is \mathcal{T} and whose dynamics on \mathcal{T} has a hyperbolic periodic orbit. In order to prove this perturbative statement, one first notice that all the Lyapunov exponents along \mathcal{T} for X vanish and \mathcal{T} is r -normally hyperbolic. By [HPS, Theorem 4.1], the set \mathcal{T} is C^r -diffeomorphic to \mathbb{T}^2 and any C^r -perturbation of $X|_{\mathcal{T}}$ extends to M . Since Morse-Smale vector fields are dense in $\mathcal{X}^r(\mathbb{T}^2)$ by [Pe], the result follows. \square

Here are some consequence of the connecting lemma for pseudo-orbits.

Theorem 8.2. *If X is generic in $\mathcal{X}^1(M)$, then for any non-trivial chain-recurrence class C containing a hyperbolic singularity σ whose unstable space is one-dimensional, C is Lyapunov stable and every separatrix of $W^u(\sigma)$ is dense in C .*

In particular if $\dim(M) = 3$, a chain-transitive set which strictly contains a singularity is a chain-recurrence class.

Proof. The first part has been shown [GY, Lemmas 3.14 and 3.19]: it is a consequence of the version for flows of the connecting lemma proved in [BC₁].

If $\dim(M) = 3$, and if Λ is a non-trivial chain-transitive set containing a singularity σ , then σ cannot be a sink, nor a source. Let us assume that σ has a one-dimensional unstable space (otherwise it has one-dimensional stable space and the proof is similar). From the first part, Λ should contain one of the separatrix of $W^u(\sigma)$. Since every separatrix of $W^u(\sigma)$ is dense in $C(\sigma)$, Λ coincides with the chain-recurrence class of σ . \square

In order to obtain the singular hyperbolicity on a chain-transitive set, it suffices to check that the tangent flow has a dominated splitting. In the non-singular case, this is proved in [BGY, Lemma 3.1] from [ARH]².

Theorem 8.3. *If $\dim(M) = 3$ and if X is generic in $\mathcal{X}^1(M)$, then for any chain-transitive set Λ such that $\Lambda \cap \text{Sing}(X) = \emptyset$, if the linear Poincaré flow on Λ has a dominated splitting, then Λ is hyperbolic.*

²Note that it could also be obtained from Theorem A' with a Baire argument.

In the singular case, this is [GY, Theorem C].

Theorem 8.4. *If $\dim(M) = 3$ and if X is generic in $\mathcal{X}^1(M)$, then any non-trivial chain-recurrence class whose tangent flow has a dominated splitting and which contains a singularity is singular hyperbolic.*

Let us summarize some properties satisfied by C^1 generic vector fields that are away from homoclinic tangencies.

Theorem 8.5. *Consider a generic $X \in \mathcal{X}^1(M)$, a non-trivial chain-recurrent class C and neighborhoods \mathcal{U}, U of X, C such that for any $Y \in \mathcal{U}$, the maximal invariant set of Y in U does not contain a homoclinic tangency of a hyperbolic periodic (regular) orbit. Then, there exists a dominated splitting on $C \setminus \text{Sing}(X)$ for the linear Poincaré flow.*

Proof. This is a variation of the arguments of [GY]. We first state a general genericity result.

Lemma 8.6. *For any generic $X \in \mathcal{X}^1(M)$, any non-trivial chain-transitive set Λ is the limit for the Hausdorff topology of a sequence of hyperbolic periodic saddles.*

Proof. Any non-trivial chain-transitive set is the limit for the Hausdorff topology of a sequence of hyperbolic periodic orbits γ_n . This has been shown in [Cr₁] for diffeomorphisms, but the proof is the same for vector fields. If there exists infinitely many γ_n that are saddles, the lemma is proved. One can thus deal with the case all the γ_n are sinks (the case of sources is similar).

By [GY, Lemma 2.23], the sinks γ_n are not uniformly contracting at the period (see the precise definition there). By [GY, Lemma 2.6], by an arbitrarily small C^1 -perturbation, one can turn the γ_n, n large, to saddles. Then by a Baire argument, one concludes that Λ is the limit of a sequence of hyperbolic periodic saddles. \square

[GY, Corollary 2.10] asserts that if Λ is the limit of a sequence of hyperbolic saddles for the Hausdorff topology, then there exists a dominated splitting for the linear Poincaré flow on $\Lambda \setminus \text{Sing}(X)$, assuming X is not accumulated by vector fields in $\mathcal{X}^1(M)$ with a homoclinic tangency. The same proof can be localized, assuming that Λ is a chain-recurrence class and that there is no homoclinic tangency in a neighborhood of Λ for vector fields C^1 -close to X . \square

We also state the result proved in [CY₂] which asserts that singular hyperbolicity implies robust transitivity for generic vector fields in dimension 3 (and improve a previous result by Morales and Pacifico [MP]).

Theorem 8.7. *If $\dim(M) = 3$ and if $X \in \mathcal{X}^1(M)$ is generic, any singular hyperbolic chain-recurrence class is robustly transitive.*

8.2 Singularities of Lyapunov stable chain-recurrence classes

The domination of the linear Poincaré flow constrains the local dynamics at singularities.

Proposition 8.8. *Assume $\dim(M) = 3$. Consider a generic $X \in \mathcal{X}^1(M)$ and a non-trivial chain-recurrence class C containing a singularity with stable dimension equal to 2 such that there exists a dominated splitting on $C \setminus \text{Sing}(X)$ for the linear Poincaré flow.*

Then, any singularity σ in C has stable dimension equal to 2, real simple eigenvalues and satisfies $W^{ss}(\sigma) \cap C = \{\sigma\}$.

Note that for any singularity σ with stable dimension equal to 2 and real simple eigenvalues, any point $x \in W^u(\sigma)$ has a well defined two-dimensional center unstable plane $E^{cu}(x)$ (it is the unique plane at x which converge to the center-unstable plane of σ by backward iterations). We will use the next lemma.

Lemma 8.9. Assume $\dim(M) = 3$. Consider any $X \in \mathcal{X}^1(M)$, any singularity with stable dimension equal to 2 and real simple eigenvalues, any $x \in W_{loc}^u(\sigma) \setminus \{\sigma\}$ satisfying $\omega(x) \cap W^{ss}(\sigma) \setminus \{\sigma\} \neq \emptyset$. There exists $\alpha > 0$ small such that for any neighborhood \mathcal{U} of X in $\mathcal{X}^1(M)$ and any $\varepsilon > 0$, there is $Y \in \mathcal{U}$ satisfying:

- $X = Y$ on $\{\varphi_{-t}(x), t \geq 0\} \cup \{\sigma\}$,
- there exists $s > 0$ such that the flow φ^Y associated to Y satisfies

$$d(\varphi_s^Y(x), x) < \varepsilon \text{ and } d(D\varphi_s^Y(x).E^{cu}(x), E^{cu}(x)) > \alpha.$$

Proof. Up to replace X by a vector field close, one can assume that:

- $x \in (W^u(\sigma) \cap W^{ss}(\sigma)) \setminus \{\sigma\}$ (using the connecting lemma [H]).
- There exists a chart on a neighborhood of σ which linearizes X : in particular, at any point z in the chart one defines the planes $H_z, V_z \subset T_z M$ which are parallel to $E^{ss}(\sigma) \oplus E^u(\sigma)$ and to $E^{cu}(\sigma)$ respectively; the flow along pieces of orbits in the chart preserves the bundle H . Moreover $E^{cu}(z) = V_z$ at points z in the local unstable manifold of σ .
- $E^{cu}(x)$ is not tangent to $T_x W^s(\sigma)$.

Let $\alpha > 0$ smaller than $d(H_\sigma, V_\sigma)$. Note that if $E \subset TM$ is a plane at a point z in the orbit of x such that $E \neq E^{cu}(z)$ and $X(z) \in E$, then $D\varphi_t.E^{cu}(z)$ converge to the bundle H when t goes to $+\infty$. This is a direct consequence of the dominated splitting between E^s and E^u .

After a small perturbation which preserves $\{\sigma\} \cup \{\varphi_t(x), t \in \mathbb{R}\}$ and $\{D\varphi_{-t}.E^{cu}(x), t > 0\}$ and whose support is disjoint from a small neighborhood of σ , one can thus assume that there exist $t_1 > 0$ arbitrarily large such that for any $t > t_1$

$$D\varphi_t.E^{cu}(x) = H_{\varphi_t(x)}.$$

Indeed after a small perturbation in a small neighborhood of $\varphi_1(x)$, and which does not change the orbit of x , one can assume that the new center-unstable space E at $\varphi_2(x)$ does not coincide with the initial one $E_{\varphi_2(x)}^{cu}$. The property mentioned above then implies that $D\varphi_t.E^{cu}(x)$ converges H_σ as t goes to $+\infty$. A new perturbation at a large iterate of x then guaranties that $D\varphi_{t_1}.E^{cu}(x) = H_{\varphi_{t_1}(x)}$.

After a small perturbation near $\varphi_{t_1}(x)$, one gets a forward iterate $\varphi_{t_2}(x)$, $t_2 > t_1$, arbitrarily close to x such that $D\varphi_{t_2}.E^{cu}(x) = H_{\varphi_{t_2}(x)}$. This implies that for the new vector field $E^{cu}(x) \subset T_x M$ has a large forward iterate arbitrarily close to $H_x \subset T_x M$ as required. \square

It has the following consequence.

Corollary 8.10. Assume $\dim(M) = 3$. For any generic $X \in \mathcal{X}^1(M)$ and any chain-recurrence class C containing a singularity σ with stable dimension equal to 2 and real simple eigenvalues such that $W^{ss}(\sigma) \cap C \neq \{\sigma\}$, there exists $x \in W_{loc}^u(\sigma) \cap C$ and $t_n \rightarrow +\infty$ such that

$$\varphi_{t_n}(x) \rightarrow x \text{ and } D\varphi_{t_n}(x).E^{cu}(x) \not\rightarrow E^{cu}(x). \quad (16)$$

Proof. For $\varepsilon, \delta, \alpha > 0$ let us consider the open property:

$$P(\varepsilon, \delta, \alpha): \exists x \in W_\delta^u(\sigma), \exists s > 0, d(\varphi_s(x), x) < \varepsilon \text{ and } d(D\varphi_s(x).E^{cu}(x), E^{cu}(x)) > \alpha.$$

Consider X is generic and a singularity σ with stable dimension equal to 2 and real simple eigenvalues. Lemma 8.9 gives $\alpha > 0$. By genericity, for any integers N_1, N_2 , one can require that if $P(1/N_1, 1/N_2, \alpha)$ holds for an arbitrarily small perturbation of X and the continuation of σ , then it holds also for X, σ .

If $W^{ss}(\sigma) \cap C \neq \{\sigma\}$, then Lemma 8.9 shows that $P(1/N_1, 1/N_2, \alpha)$ holds for any N_1, N_2 by small C^1 -perturbation. Hence X, σ satisfies $P(1/N_1, 1/N_2, \alpha)$ for any N_1, N_2 . This gives $x \in W^u(\sigma) \cap C$ and $t_n \rightarrow +\infty$ such that (16) holds. \square

Proof of Proposition 8.8. Since X is generic, one can assume that any singularity is hyperbolic. Since C contains a singularity with stable dimension equal to 2, by Theorem 8.2 it is Lyapunov stable, in particular it contains the unstable manifolds of its singularities. Let $\mathcal{N} = \mathcal{E} \oplus \mathcal{F}$ be the dominated splitting for the linear Poincaré flow on $C \setminus \text{Sing}(X)$.

Any singularity σ with stable dimension equal to 2 has real eigenvalues: indeed, by iterating backwards the dominated splitting of the linear Poincaré flow along an unstable orbit of σ , one deduces that the two stable eigenvalues have different moduli. If one assumes by contradiction that $W^{ss}(\sigma) \cap C(\sigma) \neq \{\sigma\}$, then there exists $x \in W^u(\sigma) \cap C$ and $t_n \rightarrow +\infty$ such that (16) holds. We have $\mathcal{F}(x) = E^{cu}(x) \cap \mathcal{N}_x$: indeed, for any two planes $E_1, E_2 \in T_x M$ containing $X(x)$ and different from $E^{cu}(x)$, the backward iterates converge to $E^{ss}(\sigma) \oplus E^u(\sigma)$, and get arbitrarily close; hence for any two lines $\mathcal{E}_1, \mathcal{E}_2 \subset \mathcal{N}_x$ different from $E^{cu}(x) \cap \mathcal{N}_x$, the backward iterates under the linear Poincaré flow ψ get arbitrarily close and this property characterizes the space $\mathcal{F}(x)$. Moreover \mathcal{F} is continuous at non-singular points of C (by uniqueness of the dominated splitting). But this contradicts (16) as in Corollary 8.10 which can be restated as:

$$\varphi_{t_n}(x) \rightarrow x \text{ and } \psi_{t_n}(x) \cdot \mathcal{F}(x) \not\rightarrow \mathcal{F}(x).$$

We thus have $W^{ss}(\sigma') \cap C(\sigma) = \{\sigma\}$.

Let us assume now by contradiction that C contains a singularity with stable dimension equal to 1. One can apply the previous discussions to $-X$: the class contains the stable manifold of its singularities. This contradicts the fact that $W^{ss}(\sigma) \cap C = \{\sigma\}$. Consequently any singularity in C has stable dimension equal to 2. This ends the proof. \square

8.3 Dominated splitting on singular classes

Now we can prove Theorem A. Let us consider X in the residual set of vector fields satisfying Theorems 8.1, 8.2, 8.3, 8.5, Proposition 8.8 and the following property:

If the chain-recurrence class $C(\sigma)$ of a hyperbolic singularity has no dominated splitting for the tangent flow, then for the vector fields Y that are C^1 -close to X , the tangent flow on the chain-recurrence class $C(\sigma_Y)$ of the continuation of X has no dominated splitting.

This property can be deduced from the semi-continuity of $Y \mapsto C(\sigma_Y)$ and the fact that if the tangent flow on an invariant compact set Λ_0 for Y_0 is dominated, it is still the case for any compact set Λ Hausdorff close to Λ_0 and vector fields Y C^1 -close to Y_0 .

Let Λ be a chain-transitive set of X such that the tangent flow on Λ has a dominated splitting $E \oplus F$. If Λ is non-singular, let us consider the two disjoint invariant compact sets $K_E := \{x, X(x) \in E\}$ and $K_F := \{x, X(x) \in F\}$. For $\varepsilon > 0$ let $A := \{x, d(X(x), E) \geq \varepsilon\}$. By domination, the set A is sent into its interior by large forward iterates. The chain-transitivity implies that $\Lambda = A$ or $A = \emptyset$. Since the orbit of any point $x \notin K_E \cup K_F$ accumulates to K_F in the future and to K_E in the past, this gives $\Lambda = K_E$ or $\Lambda = K_F$. Without loss of generality we assume the first case. Then $\dim(E) = 2$, since otherwise F is uniformly expanded by the domination and Λ is a source. Thus for any $x \in \Lambda$, $\mathcal{E}(x) := E(x) \cap \mathcal{N}_x$ and $\mathcal{F}(x) := (F(x) \oplus \mathbb{R}X(x)) \cap \mathcal{N}_x$ are two one-dimensional lines which define two bundles invariant and dominated under ψ .

In the remaining case, Λ contains a singularity σ . By Theorem 8.2, Λ is a chain-recurrence class. The existence of a dominated splitting is an open property: there are neighborhoods \mathcal{U} , U of X , Λ such that for any $Y \in \mathcal{U}$, the maximal invariant set of Y in U has a dominated splitting; in particular, it does not contain a homoclinic tangency of a periodic orbit. Hence the assumptions of Theorem 8.5 hold and the linear Poincaré flow on $\Lambda \setminus \text{Sing}(X)$ is dominated. This proves one half of Theorem A.

We now assume Λ is a chain-transitive set of X such that the linear Poincaré flow on $\Lambda \setminus \text{Sing}(X)$ has a dominated splitting. If Λ does not contain any singularity, it is a hyperbolic set by Theorem 8.3: hence it has a dominated splitting also. Thus we assume that Λ contains a singularity σ with stable dimension equal to 2 (the case of dimension 1 is similar). By Theorem 8.2, Λ is a chain-recurrence class and is Lyapunov stable.

By Proposition 8.8, any singularity $\tilde{\sigma} \in \Lambda$ has stable dimension equal to 2, real simple eigenvalues and satisfies $W^{ss}(\tilde{\sigma}) \cap \Lambda = \{\tilde{\sigma}\}$. Note that these properties still hold and the linear Poincaré flow is still dominated for the vector fields C^1 -close and the chain-recurrence class of the continuation of σ : one uses that the chain-recurrence class of the continuation of σ vary semi-continuously with the vector field X , that the set of singularities is finite and Proposition 2.8. We consider a vector field Y that is C^1 -close to X , and that is C^2 , whose regular periodic orbits are hyperbolic and with no normally expanded invariant torus whose dynamics is topologically equivalent to an irrational flow (see Theorem 8.1 in $\mathcal{X}^3(M)$). In particular the periodic orbits in the chain-recurrence class $C(\sigma_Y)$ of the continuation of σ for Y are neither sources nor sinks and have a negative Lyapunov exponent. One can thus apply Theorem A': there exists a dominated splitting for the tangent flow on $C(\sigma_Y)$. By our choice of the generic vector field X , this is also the case for the chain-recurrence class of σ for X , which is the set Λ .

The Theorem A is proved. \square

8.4 Dichotomy for three-dimensional vector fields

We now complete the proofs of the Main Theorem and of Corollary 1.1.

Proof of the Main Theorem. Let us consider a vector field X in the intersection of the residual sets provided by Theorems A, 8.1, 8.3, 8.4 and 8.5. Let us assume that it can not be accumulated in $\mathcal{X}^1(M)$ by vector fields with homoclinic tangencies.

Let C be a chain-recurrence class of X . By Theorem 8.1, if C is an isolated singularity or a regular periodic orbit, it is hyperbolic. If C is non-trivial, by Theorem 8.5 the linear Poincaré flow on $C \setminus \text{Sing}(X)$ is dominated. Using Theorem 8.3 if $C \cap \text{Sing}(X) = \emptyset$, the class C is hyperbolic. In the remaining case, C is non-trivial, contains a singularity, the tangent flow on C is dominated (by Theorem A). Hence C is singular hyperbolic (by Theorem 8.4).

Since any chain-recurrence class of X is singular hyperbolic, X is singular hyperbolic. \square

Proof of Corollary 1.1. Let \mathcal{O} be the set of C^1 vector fields on M whose chain-recurrence classes are robustly transitive. We then introduce the dense set $\mathcal{U} = \mathcal{O} \cup (\mathcal{X}^1(M) \setminus \overline{\mathcal{O}})$.

We claim that \mathcal{O} (and thus \mathcal{U}) is open. Indeed for $X \in \mathcal{O}$, each chain-recurrence class is isolated in the chain-recurrent set: let us consider a class C ; by semi-continuity of the chain-recurrence classes for the Hausdorff topology, if C' is another class having a point close to C , it is contained in a small neighborhood of C , hence coincide with C by definition of the robust transitivity. This implies that X has only finitely many chain-recurrence classes C_1, \dots, C_k . By robust transitivity, each of them admits a neighborhood U_1, \dots, U_k so that for any Y close to X in $\mathcal{X}^1(M)$, the maximal invariant set in each U_i is robustly transitive. By semi-continuity of the chain-recurrence classes, each class of Y has to be contained in one of the U_i , hence is robustly transitive, as required.

Let \mathcal{G} be the dense G_δ set of vector fields in $\mathcal{X}^1(M)$ such that Theorem 8.7 holds. Let us consider any $X \in \mathcal{U}$ that can not be approximated by vector fields exhibiting a homoclinic tangency. By the Main Theorem, there exists X' arbitrarily close to X in $\mathcal{X}^1(M)$ which is singular hyperbolic. Since singular hyperbolicity is an open property and \mathcal{G} is dense, one can also require that $X' \in \mathcal{G}$, hence each chain-recurrence class of X is robustly transitive. We have thus shown that $X \in \overline{\mathcal{O}}$. By definition of \mathcal{U} this gives $X \in \mathcal{O}$ and the Corollary follows. \square

References

- [ABC] F. Abdenur, C. Bonatti and S. Crovisier, Non-uniform hyperbolicity for C^1 -generic diffeomorphisms. *Israel J. Math.* **183** (2008), 1–60.
- [ABS] V. Afraïmovič, V. Bykov, and L. Silnikov, The origin and structure of the Lorenz attractor. *Dokl. Akad. Nauk SSSR* **234** (1977), 336–339.
- [ARH] A. Arroyo and F. Rodriguez Hertz, Homoclinic bifurcations and uniform hyperbolicity for three-dimensional flows. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **20** (2003), 805–841.
- [AP] V. Araújo and M.J. Pacifico, Three-dimensional flows. *Ergebnisse der Mathematik und ihrer Grenzgebiete* **53**. Springer (2010).
- [BC₁] C. Bonatti and S. Crovisier, Récurrence et genericité. *Invent. Math.* **158** (2004), 33–104.
- [BC₂] C. Bonatti and S. Crovisier, Center manifolds for partially hyperbolic set without strong unstable connections. *Journal of the IMJ* **15** (2016), 785–828 .
- [BDV] C. Bonatti L. Díaz and M. Viana, Dynamics beyond uniform hyperbolicity. A global geometric and probabilistic perspective. *Encyclopaedia of Mathematical Sciences* **102**. Mathematical Physics, III. Springer-Verlag (2005).
- [BGY] C. Bonatti, S. Gan and D. Yang, Dominated chain recurrent classes with singularities. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **14** (2015), 83–99.
- [Co] C. Conley, *Isolated invariant sets and Morse index*. CBMS Regional Conference Series in Mathematics **38**. AMS (1978).
- [Cr₁] S. Crovisier, Periodic orbits and chain-transitive sets of C^1 -diffeomorphisms. *Publ. Math. Inst. Hautes Études Sci.* **104** (2006), 87–141.
- [Cr₂] S. Crovisier, Perturbation de la dynamique de difféomorphismes en topologie C^1 . *Astérisque* **354** (2013).
- [CP] S. Crovisier and E. Pujals, Essential hyperbolicity and homoclinic bifurcations: a dichotomy phenomenon/mechanism for diffeomorphisms. *Invent. Math.* **201** (2015), 385–517.
- [CPS] S. Crovisier, E. Pujals and M. Sambarino, *Hyperbolicity of the extremal bundles*. In preparation.
- [CY₁] S. Crovisier and D. Yang, On the density of singular hyperbolic three-dimensional vector fields: a conjecture of Palis. *C. R. Math. Acad. Sci. Paris* **353** (2015), 85–88.
- [CY₂] S. Crovisier and D. Yang, *Robust transitivity of singular hyperbolic attractors*. In preparation.
- [dMvS] W. de Melo and S. van Strien, *One-dimensional dynamics*. Ergebnisse der Mathematik und ihrer Grenzgebiete **25**. Springer-Verlag (1993).
- [GY] S. Gan and D. Yang, *Morse-Smale systems and horseshoes for three dimensional flows*. To appear at *Ann. Sci. École Norm. Sup.* Preprint (2013) arXiv:1302.0946.
- [G] J. Guckenheimer, A strange, strange attractor. *The Hopf bifurcation theorems and its applications. Applied Mathematical Series* **19**. Springer-Verlag (1976), 368–381.
- [GW] J. Guckenheimer and R. Williams, Structural stability of Lorenz attractors. *Publ. Math. Inst. Hautes Études Sci.* **50** (1979), 59–72.
- [H] S. Hayashi, Connecting invariant manifolds and the solution of the C^1 -stability and Ω -stability conjectures for flows. *Ann. of Math.* **145** (1997), 81–137 and *Ann. of Math.* **150** (1999), 353–356.

- [HH] G. Hector and U. Hirsch, *Introduction to the geometry of foliations*. Vieweg (1981).
- [HPS] M. Hirsch, C. Pugh and M. Shub, *Invariant manifolds*. Lecture Notes in Mathematics **583** (1977).
- [K] I. Kupka. Contributions à la théorie des champs génériques. *Contrib. Differential Equations* **2** (1963), 457–484.
- [LGW] M. Li, S. Gan and L. Wen, Robustly transitive singular sets via approach of extended linear Poincaré flow. *Discrete Contin. Dyn. Syst.* **13** (2005), 239–269.
- [Li₁] S. Liao, Certain ergodic properties of a differential system on a compact differentiable manifold. *Acta Sci. Natur. Univ. Pekinensis* **9** (1963), 241–265 and 309–326. Translated in *Front. Math. China* **1** (2006), 1–52.
- [Li₂] S. Liao, On (η, d) -contractible orbits of vector fields. *Systems Science and Mathematical Sciences* **2** (1989), 193–227.
- [Lo] E. N. Lorenz, Deterministic nonperiodic flow. *J. Atmosph. Sci.* **20** (1963), 130–141.
- [M₁] R. Mañé, Hyperbolicity, sinks and measure in one-dimensional dynamics. *Comm. Math. Phys.*, **100** (1985), 495–524.
- [M₂] R. Mañé, *Ergodic Theory and Differential Dynamics*. Springer-Verlag (1987).
- [MCY] Z. Mi, Y. Cao and D. Yang, *SRB measures for diffeomorphisms with continuous invariant splittings*. Preprint (2015) arXiv:1508.02747.
- [MP] C. Morales and M. Pacifico, A dichotomy for three-dimensional vector fields. *Ergodic Theory and Dynamical Systems* **23** (2003), 1575–1600.
- [MPP] C. Morales, M. Pacifico, and E. Pujals, Robust transitive singular sets for 3-flows are partially hyperbolic attractors or repellers. *Ann. of Math.* **160** (2004), 375–432.
- [Ne] S. Newhouse, Nondensity of axiom A(a) on S^2 . *Global analysis I, Proc. Symp. Pure Math. AMS* **14** (1970), 191–202.
- [No] A. Nobile, Some properties of the Nash blowing-up. *Pacific J. Math.* **60** (1975), 297–305.
- [Pa₁] J. Palis, Homoclinic bifurcations, sensitive-chaotic dynamics and strange attractors. in *Dynamical systems and related topics* (Nagoya, 1990). *Adv. Ser. Dynam. Systems* **9**. World Sci. Publishing (1991), 466–472.
- [Pa₂] J. Palis, A global view of dynamics and a conjecture of the denseness of finitude of attractors. *Astérisque*, **261** (2000), 335–347.
- [Pa₃] J. Palis, A global perspective for non-conservative dynamics. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **22** (2005), 485–507.
- [Pa₄] J. Palis, Open questions leading to a global perspective in dynamics. *Nonlinearity*, **21** (2008), 37–43.
- [Pe] M. Peixoto, Structural stability on two-dimensional manifolds. *Topology* **1** (1962), 101–120.
- [PS₁] E. Pujals and M. Sambarino, Homoclinic tangencies and hyperbolicity for surface diffeomorphisms. *Ann. of Math.* **151** (2000), 961–1023.
- [PS₂] E. Pujals and M. Sambarino, Integrability on codimension one dominated splitting. *Bull. Braz. Math. Soc.* **38** (2007), 1–19.
- [Pu] C. Pugh, Structural stability on M . *An. Acad. Brasil. Ci.* **39** (1967), 45–48.
- [S] S. Smale, Stable manifolds for differential equations and diffeomorphisms. *Ann. Scuola Norm. Sup. Pisa* **17** (1963), 97–116.
- [T] F. Takens, Singularities of vector fields. *Publ. Math. Inst. Hautes Études Sci.* **43** (1974), 47–100.

Sylvain Crovisier

Laboratoire de Mathématiques d'Orsay
CNRS - Université Paris-Sud
Orsay 91405, France
`Sylvain.Crovisier@math.u-psud.fr`

Dawei Yang

School of Mathematical Sciences
Soochow University
Suzhou, 215006, P.R. China
`yangdw1981@gmail.com`, `yangdw@suda.edu.cn`